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Bosonization of Fermi systems in arbitrary dimension in terms of gauge forms

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Abstract. We present a general method to bosonize systems of fermions with infinitely many degrees of freedom, in particular systems of non-relativistic electrons at positive density, by expressing the quantized conserved electric charge- and current density in terms of a bosonic antisymmetric tensor field of rank $d-1$, where d is the dimension of space. This enables us to make concepts and tools from gauge theory available for the purpose of analysing electronic structure of non-relativistic matter. We apply our bosonization identities and concepts from gauge theory, such as the Wegner–'t Hooft duality, to a variety of systems of condensed matter physics: Landau–Fermi liquids, Hall fluids, London superconductors, etc. Among our results are an exact formula for the plasmon gap in a metal, a simple derivation of the Anderson–Higgs mechanism in superconductors, and an analysis of the orthogonality catastrophe for static sources.

1. Introduction and summary of main results

In this paper, we develop a conceptual framework, based on bosonization of quantum systems with infinitely many degrees of freedom, which we expect to be useful in attempts to classify states of non-relativistic matter at very low temperatures. In this paper, we focus our attention on the analysis of electronic structure. Magnetic properties will be discussed in a separate paper.

The basic ideas underlying our approach are very simple: our starting point is to study the response of a quantum system of charged particles to perturbations by external electromagnetic fields. Thus we couple the electric current density to an arbitrary, smooth external electromagnetic vector potential, A, and then attempt to calculate the partition function, $\Xi(A)$, of the system as a functional of A.

Of course, this is a very complicated task. However, in order to classify electronic structure of non-relativistic matter, we are really only interested in understanding the behaviour of the *effective action*

$$S(A) \equiv -i\hbar \ln \Xi(A) \tag{1.1}$$

on very large distance scales and at very low frequencies. We thus study families of systems confined to ever larger cubes, $\Omega_\lambda := \{x : \frac{x}{\lambda} \in \Omega\}$, in physical space $\mathbb{R}^d, d = 1, 2, 3$ where

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Ω is a fixed cube in $\mathbb{R}^d$ and $1 \leqslant \lambda < \infty$ is a scale parameter. We keep the particle density, ρ, and the temperature $T(\approx 0)$ constant. We then couple the electric current density of the system confined to Ω_λ to a vector potential $A^{(\lambda)}$ given by

$$A^{(\lambda)}(t, x) \equiv \lambda^{-1} A\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \qquad \frac{x}{\lambda} \in \Omega$$

where A is an arbitrary , but λ-independent vector potential on space-time $\mathbb{R} \times \Omega$. We then study the behaviour of $S^{\Omega_\lambda}(A^{(\lambda)})$ when λ becomes large.

More precisely, we attempt to expand $S^{\Omega_\lambda}(A^{(\lambda)})$ in powers of λ^{-1} around $\lambda = \infty$ (to a finite order) and define the *scaling limit*, $S^\star(A)$, of the effective action to be the *coefficient of the leading power* of λ in that expansion, (in the limit when $\Omega_\lambda \nearrow \mathbb{R}^d$); see [1].

One of our main contentions in this paper is the conjecture that $S^\star(A)$ is *quadratic* in A, for all systems of non-relativistic electrons at positive density. This conjecture can be verified for several systems, in particular for insulators, Landau Fermi liquids, metals, (incompressible) quantum Hall fluids, superconductors.

Remark. It is also interesting to study the response of systems of condensed matter to coupling the spin current to an external $SU(2)$ non-Abelian gauge field, W. One may then try to determine the scaling limit of the effective action of such systems as a functional of W. It is a simple consequence of $SU(2)$ gauge-invariance that $S^\star(W)$ is *not* quadratic in W. For a special class of systems, $S^\star(W)$ has been calculated in [1], and the results in that paper confirm our claim.

We do not know a proof of the conjecture described above that covers all imaginable systems. We thus rely on a case-by-case analysis. The details of our analysis will appear in separate papers [2]. At this point it should be emphasized that, except in the case of insulators and incompressible quantum Hall fluids, the proof of the conjecture is never trivial and does *not* simply follow from dimensional analysis.

It is well know that $S(A)$ is the generating functional of connected Green functions of the electric current density and that it is *gauge-invariant*, i.e.

$$S(A + d\chi) = S(A) \tag{1.2}$$

where χ is an arbitrary real function and $d\chi$ is its gradient†. Clearly, gauge-invariance persists up on passing to the scaling limit. Using our conjecture, we conclude that

$$S^\star(A) = \tfrac{1}{2}(A, \Pi^\star A) = \tfrac{1}{2}\int d^{d+1}x \int d^{d+1}y\; A_\mu(x)\Pi^{\star\mu\nu}(x-y)A_\nu(y) \tag{1.3}$$

where, for $x \neq y$, $\Pi^{\star\mu\nu}(x-y)$ is given by the scaling limit of the two-current Green function $\langle T(\mathcal{J}^\mu(x)\mathcal{J}^\nu(y))\rangle^c$.

By gauge invariance, or, equivalently, current conservation

$$\partial_\mu \Pi^{\star\mu\nu} = \partial_\nu \Pi^{\star\mu\nu} = 0\,. \tag{1.4}$$

Furthermore, $\Pi^{\star\mu\nu}$ inherits all the symmetries of the system; (in (1.3), we have assumed translation invariance). Thus, classifying electronic structure of a system reduces, in the

† We shall use the notation and elementary concepts of Cartan's exterior calculus throughout this paper.

scaling limit and under the assumption that our conjecture holds, to a classification of 'vacuum polarisation tensors', $\Pi^{*\mu\nu}$, satisfying (1.4) and having certain symmetries. This is a straightforward task; see section 5.

In principle, essentially all information concerning electronic properties of a system can be retrieved from its effective action $S(A)$ by fairly straightforward calculations. However, in order to evoke and then apply analogies with other physical systems, in particular with gauge theories of elementary particle physics, it is useful to embark on a detour. The detour chosen in this paper is *bosonization.* The idea (see [3]) is as follows: since the electric current density $\mathcal{J} = (\mathcal{J}^\mu)_{\mu=0}^d$ is conserved, it can be derived from a 'potential'

$$\mathcal{J}^\mu = \epsilon^{\mu\mu_1\cdots\mu_d}\partial_{\mu_1} b_{\mu_2\cdots\mu_d} \tag{1.5}$$

where ϵ is the totally antisymmetric ϵ-tensor and $b_{\mu_2,\ldots,\mu_d}$ is an antisymmetric tensor field of rank $d-1$. In the language of differential forms, equation (1.5) is expressed as

$$\mathcal{J} = {}^*db \tag{1.6}$$

where $*$ is the Hodge $*$ operation and d denotes exterior differentiation. The 'potential' b of the current density is determined by (1.6) only up to the exterior derivation of an antisymmetric tensor field of rank $d-2$, (a $(d-2)$-form). Thus b is what one calls a *'gauge form'*.

For one-dimensional systems, b is a scalar and is determined by (1.6) up to a constant. In two dimensions, b is a 1-form determined by $\mathcal{J}$ up to the exterior derivative of an arbitrary 0-form.

The basic idea is then to deduce an effective field theory for the field b from $S(A)$. This field theory is described in terms of an action $\tilde{S}(b)$. Choosing units in which $\hbar = 1$ and using an imaginary-time (Euclidean) formulation, $\tilde{S}(b)$ is obtained from $S(A)$ by functional Fourier transformation

$$\mathrm{e}^{-\tilde{S}(b)} = N^{-1}\int \mathrm{e}^{-S(A)}\exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge db\right)\mathcal{D}A \tag{1.7}$$

where N is a (divergent) normalization factor (proportional to the volume of the Lie algebra of $U(1)$-gauge transformations). Gauge invariance implies that

$$\tilde{S}(b + d\Lambda) = \tilde{S}(b) \tag{1.8}$$

for an arbitrary $(d-2)$-form Λ $(d \geqslant 2)$.

Our conjecture then implies that the low-wavevector, low-energy modes of the *bosonic* field b are *non-interacting*, i.e. the scaling limit of the system described in terms of the b-field has a *quadratic* action, $\tilde{S}^*(b)$, whose form is constrained by its gauge invariance, equation (1.8), and by the symmetries of the system.

Essentially all quantities of interest in the original system can be expressed in terms of quantities referring to the b-field. For example, current Green functions (at imaginary time) are given by expectations of products of the 'field strength' db in the functional measure

$$\Xi^{-1}\mathrm{e}^{-\tilde{S}(b)}\mathcal{D}b\,.$$

The Green functions of electron creation and annihilation operators turn out to be proportional to expectations of *disorder operators* of the dual theory formulated in terms of the b-field.

If Σ is an open subset of physical space, and $\mathcal{Q}_\Sigma$ denotes the operator measuring the total electric charge inside Σ then

$$\exp(\mathrm{i}\alpha\mathcal{Q}_\Sigma) \propto \exp\left(\mathrm{i}\alpha\int_{\partial\Sigma} b\right) \qquad \alpha\in\mathbb{R} \tag{1.9}$$

i.e. the charge operator can be reconstructed from operators analogous to the *Wilson loops* of gauge theory. The operators in (1.9) are *dual* to the disorder operators describing electron creation and annihilation, in the sense of the Wegner–'t Hooft duality, [4, 5]. This suggests that we can carry over the consequences of the 't Hooft duality from gauge theory to condensed matter physics. As a consequence we mention that if, at zero temperature, the total electric charge operator, $\mathcal{Q} =$ '$\lim_{\Sigma\nearrow\mathbb{R}^d}\mathcal{Q}_\Sigma$' is well defined on the Hilbert space of all physical states of the system then disorder Green functions, e.g. the Green function of an electron creation and annihilation operator, exhibit strong spatial cluster decomposition properties, and vice versa. This applies to insulators, incompressible Hall fluids and superconductors with (unscreened) Coulomb two-body repulsion. Furthermore, if the field b couples the ground state of the system to a massless quasiparticle then the total charge operator does not exist, because charge fluctuations are divergent in the thermodynamic limit, as in metals and massless superconductors.

For *two-dimensional* systems, both, A and b, are 1-forms defined up to gradients of scalar functions. In this case, equation (1.7) enables us to define a concept of *duality*: a system 1 and a system 2 are *dual* to each other iff

$$\tilde{S}_1^\star \propto S_2^\star . \tag{1.10}$$

It turns out that, in the sense of (1.10), a two-dimensional *insulator* is *dual* to a two-dimensional *London superconductor*, a metal to a 'semi-conductor', and an incompressible Laughin Hall fluid is *self-dual*.

Besides the duality expressed in (1.7) and (1.10) there is also a concept of Kramers–Wannier duality: in *any dimension* d, a $U(1)$-gauge theory of an antisymmetric tensor field, b, of rank $d-1$ is 'Kramers–Wannier-dual' to a *scalar* field theory. Thus, for example, a London superconductor, corrected by dynamical Abrikosov vortices, is Kramers–Wannier dual to a Landau–Ginsburg superconductor; see [1].

The two concepts of duality sketched here are conceptually quite clarifying and useful in a classification of electronic properties of non-relativistic matter. One of the principal advantages of reformulating the theory of a system of electrons in terms of the tensor field b (bosonization) is that this formulation is convenient to explore systems obtained by perturbing a given one by *two-body interactions*. A translation-invariant two-body interaction, I_{pert}, has the form

$$I_{\mathrm{pert}} = \int d^{d+1}x \int d^{d+1}y\; \mathcal{J}^\mu(x)V_{\mu\nu}(x-y)\mathcal{J}^\nu(y) . \tag{1.11}$$

After bosonization, the action of the perturbed system is given by

$$\tilde{S}_{tot}(b) = \tilde{S}(b) + \int d^{d+1}x \int d^{d+1}y\; ({}^*db)^\mu(x)V_{\mu\nu}(x-y)({}^*db)^\nu(y) \tag{1.12}$$

where $\tilde{S}(b)$ is the action of the unperturbed system. Note that, expressed in terms of the field b the two-body interaction I_{pert} is *quadratic* (rather than quartic)! A conventional two-body interaction described by an istantaneous two-body potential corresponds to a kernel $V_{\mu\nu}$ given by

$$V_{\mu\nu}(x-y) = \delta_{\mu 0}\delta_{\nu 0}V(\boldsymbol{x}-\boldsymbol{y})\delta(x^0-y^0) \tag{1.13}$$

with $x = (x^0, \boldsymbol{x})$, $y = (y^0, \boldsymbol{y})$.

Suppose now that the action of the perturbed system in the scaling limit (scale parameter $\lambda \to \infty$), $\tilde{S}^{\star}_{\text{tot}}$, is given by the scaling limit, $\tilde{S}^{\star}$, of the action of the unperturbed system, perturbed by the long-range tail, $I^{\star}_{\text{pert}}$, of the two-body interaction I_{pert}. From our conjecture, we infer that $\tilde{S}^{\star}$ is quadratic in b, and hence, since $I^{\star}_{\text{pert}}$ is quadratic in b, $\tilde{S}^{\star}_{\text{tot}}$ is *quadratic* in b, too, and, by (1.8) and (1.12), gauge-invariant. It is given by

$$\tilde{S}^{\star}_{\text{tot}} \sim \tilde{S}^{\star} + \lambda^{\kappa} I^{\star}_{\text{pert}} \qquad (\lambda \to \infty) \tag{1.14}$$

for some exponent $\kappa \geqslant 0$. It is plausible that the assumption that perturbation by I and passage to the scaling limit are commuting operations is justified if $V_{\mu\upsilon}$ is *positive-definite*, of very *long range* and the 'Cooper channel' is turned off. In this case, our analysis yields the 'random phase approximation' (RPA). We apply these ideas to the following systems.

(i) A metal perturbed by repulsive two-body Coulomb interactions. In this case we obtain the exact formula for the *plasmon gap*.
(ii) A massless London superconductor perturbed by repulsive two-body Coulomb interactions. In this example we recover a precise formulation of the *Anderson–Higgs mechanism*.
(iii) A Landau–Fermi liquid perturbed by repulsive two-body interactions, in the presence of a static source. For a two-dimensional system of this type with Coulomb–Ampère interaction we discuss a possible cross over to a non-Fermi liquid behaviour (the *'Luttinger liquid'*).

We conclude this introduction with a brief summary of the contents of this paper.

In section 2, we derive our general *bosonization method*, based on equations (1.2), (1.6), (1.7) and (1.8), and present the main identities it provides.

In section 3, we introduce *disorder fields* of the bosonized theory and the *local electric charge operators* $\mathcal{Q}_\Sigma$. We then explain the connection between disorder fields and electron creation- and annihilation operators. We recall what is meant by 't Hooft duality and discuss its implications.

As an application of the general theory developed in sections 2 and 3 we briefly discuss in section 4 systems of relativistic massless fermions in one space dimension. In this case, we recover the standard identities of one-dimensional, Abelian bosonization. In condensed matter physics, these systems describe Landau and Luttinger Fermi liquids and can be studied using techniques from chiral $U(1)$-current algebra.

In one dimension, it is not hard to extend our methods to an analysis of magnetic properties. This is done by coupling the spin degrees of freedom of the electrons to external non-Abelian gauge fields with gauge group given by $SU(2)$, see [1]. Applying our bosonization methods to this special case, we would recover the formulae of non-Abelian bosonization and of $SU(2)$-current algebra, but we shall not present these results (which actually have just appeared in a recent preprint by Burgess and Quevedo [6]).

In section 5, we apply our methods to systems from condensed matter physics. We consider Landau–Fermi (non-interacting electron) liquids, in which case we show how to derive the Luther–Haldane bosonization formulae from our methods, insulators, incompressible Hall fluids (Laughlin fluids), and massless London superconductors. We then discuss duality, in the sense of equation (1.10), for two-dimensional systems and conclude with an analysis of Laughlin fluids.

In section 6, we consider perturbations of the systems discussed in section 5 by repulsive Coulomb(–Ampère) two-body interactions, along the lines sketched in equations (1.11)–(1.14). We find the exact expression for the plasmon gap in a metal, recover the consequences of the Anderson–Higgs mechanism in a precise form and discuss the 'orthogonality catastrophe' for static sources.

In the appendix, we outline the theory of *gauge forms*, of which the potential b of the conserved electric current density $\mathcal{J}$ (see equations (1.5) and (1.6)) is a special case.

2. Bosonization

In this section we present the details of our method of bosonization.

In the Euclidean path-integral formalism, fermions of spin S are described in terms of Grassmann fields Ψ_α, Ψ^*_α, $\alpha = 1, \ldots, 2S+1$.

We consider a system of fermions whose Euclidean action, $I(\Psi, \Psi^*)$, is local in Ψ, Ψ^* and their derivatives and invariant under the global $U(1)$-gauge tranformations

$$\Psi_\alpha(x) \longrightarrow \mathrm{e}^{\mathrm{i}\Lambda}\Psi_\alpha(x) \qquad \Psi^*_\alpha(x) \longrightarrow \mathrm{e}^{-\mathrm{i}\Lambda}\Psi^*_\alpha(x) \qquad \Lambda \in \mathbb{R}. \tag{2.1}$$

We couple the system to a $U(1)$-gauge field, A_μ, by replacing derivatives by covariant derivatives, thus gauging the symmetry (2.1). Let $I(\Psi, \Psi^*, A)$ denote the corresponding gauge-invariant action. We define the effective action, $S(A)$, of the system by setting

$$\mathrm{e}^{-S(A)} \equiv \int \mathcal{D}\Psi\, \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)}. \tag{2.2}$$

Let j be a 1-form. The Fourier transform of (2.2) is given by the following equation:

$$\mathrm{e}^{-\tilde{S}(j)} \equiv \int \mathcal{D}A\, \mathrm{e}^{-S(A)} \exp\left(\mathrm{i}\int (A_\mu(x) j^\mu(x)\, d^{(d+1)}x)\right). \tag{2.3}$$

Using the invariance of $I(\Psi, \Psi^*, A)$ under the gauge transformation

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x) \tag{2.4}$$

one can integrate the r.h.s. of (2.3) over all gauge transformations (2.4). We then obtain the constraint

$$\partial_\mu j^\mu(x) = 0 \tag{2.5}$$

i.e. a continuity equation for j. In order to simplify the notation it is useful to introduce the concept of differential forms.

Let M be an open subset of $\mathbb{R}^{d+1}$. Given an antisymmetric tensor field of rank k on M, one defines locally the associated differential form of rank k, or simply k-form, by setting

$$a^{(k)}(x) = \frac{1}{k!} a_{\mu_1 \cdots \mu_k}(x) dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \tag{2.6}$$

where $\wedge$ is the wedge (antisymmetric tensor) product. The space of k-forms is a group, $\Lambda^k(M)$, under the operation of pointwise addition. We denote by $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ the exterior differential, by $* : \Lambda^k(M) \to \Lambda^{d+1-k}(M)$ the Hodge star, by $\delta = {}^*d^*(-1)^{(d+1)(k+1)+1}$ the codifferential and by $H^k_{\mathrm{deR}}(M) = \ker(d\lceil\Lambda^k(M))/\operatorname{im}(d\lceil\Lambda^{k-1}(M))$ the kth (de-Rham) cohomology group of M (see, for example, [9] for details).

An inner product between k-forms is defined by setting

$$(a^{(k)}, b^{(k)}) = \int d^{d+1}x \, a_{\mu_1 \cdots \mu_k}(x) b^{\mu_1 \cdots \mu_k}(x) = \int a^{(k)} \wedge {}^*b^{(k)} . \tag{2.7}$$

and its satisfies

$$(a^{(k)}, db^{(k-1)}) = (\delta a^{(k)}, b^{(k-1)}) . \tag{2.8}$$

Then, according to the Poincaré lemma, equation (2.5) can be explicitly solved by introducing a $(d-1)$-form $b \equiv b^{(d-1)}$ with

$$j = \frac{1}{2\pi} \, {}^*db \tag{2.9}$$

where the factor $\frac{1}{2\pi}$ has been inserted for later convenience. For $d > 1$, the solution, b, of (2.9) is not unique, since two forms, b, b', differing by a term $d\lambda^{(d-2)}$ yields the same j, thanks to the property $d^2 = 0$. As a consequence, the action $\tilde{S}(j) \equiv \tilde{S}(db)$ of (2.3) is invariant under the gauge transformation

$$b \to b + d\lambda^{(d-2)} . \tag{2.10}$$

Remark 2.1. Equation (2.10) is the action of the gauge group $\mathcal{G}^{d-2}$, on the space of generalized connections $\mathcal{A}^{d-1}$, with $\lambda^{(d-1)}$ globally defined because $H^{d-1}_{\text{deR}}(M) = 0$ (see appendix). Thus b should be viewed as a gauge form of rank $d-1$ which is globally defined because $H^{d}_{\text{deR}}(M) = 0$.

(For later purposes, we use a notation, $\tilde{S}(db)$, slightly different from the one adopted in the introduction, i.e. $\tilde{S}(b)$.)

In terms of the $(d-1)$-form b, we may rewrite (2.3) as

$$e^{-\tilde{S}(db)} \equiv \int \mathcal{D}[A] \, e^{-S(A)} \exp\left(\frac{1}{2\pi} \int A \wedge db\right) \tag{2.11}$$

where $\mathcal{D}[A] e^{-S(A)}(\cdot)$ denotes the measure induced by $\mathcal{D}A e^{-S(A)}$ on the space of gauge orbits

$$[A] = \{A' : A - A' = d\Lambda\} . \tag{2.12}$$

Next, we wish to prove a set of bosonization identities. Let $\mathcal{D}[b] e^{-\tilde{S}(db)}$ denote the measure on the gauge equivalence classes

$$[b] = \{b' : b' = b + d\lambda^{(d-2)}\} . \tag{2.13}$$

Formally,
(1)

$$\Xi \equiv \int \mathcal{D}\Psi \, \mathcal{D}\Psi^* \, e^{-I(\Psi, \Psi^*)} = \int \mathcal{D}[b] \, e^{-\tilde{S}(db)} . \tag{2.14}$$

Let

$$\mathcal{J}^\mu(\Psi, \Psi^*, A; x) = -\mathrm{i} \frac{\delta}{\delta A_\mu(x)} I(\Psi, \Psi^*, A) \tag{2.15}$$

so that $\mathcal{J}^{\mu}(\Psi, \Psi^*, 0) \equiv \mathcal{J}^{\mu}(\Psi, \Psi^*)$ is the $U(1)$-current of the fermion system corresponding to the global symmetry (2.1).

Then, for non-coinciding points $\{x_1, \ldots, x_n\}$, we have that

(2)

$$\langle \mathcal{J}^{\mu_1}(\Psi, \Psi^*; x_1) \cdots \mathcal{J}^{\mu_n}(\Psi, \Psi^*; x_n) \rangle$$

$$\equiv \frac{1}{\Xi} \int \mathcal{D}\Psi\, \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*)} \prod_{i=1}^{n} \mathcal{J}^{\mu_i}(\Psi, \Psi^*; x_i)$$

$$= \frac{1}{\Xi} \int \mathcal{D}[b]\, \mathrm{e}^{-\tilde{S}(db)} \prod_{i=1}^{n} \frac{1}{2\pi} ({}^*db)^{\mu_i}(x_i)$$

$$\equiv \left\langle \prod_{i=1}^{n} \frac{1}{2\pi} ({}^*db)^{\mu_i}(x_i) \right\rangle . \tag{2.16}$$

Proof. Assuming one can interchange the order of integration we have, formally, that

(1)

$$\int \mathcal{D}[b]\, \mathrm{e}^{-\tilde{S}(db)} = \int \mathcal{D}[b] \int \mathcal{D}[A] \int \mathcal{D}\Psi,\ \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)} \exp\left(\frac{\mathrm{i}}{2\pi} \int A \wedge db \right)$$

$$= \int \mathcal{D}[A] \int \mathcal{D}\Psi \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)} \delta(dA)$$

$$= \int \mathcal{D}\Psi\, \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*)} . \tag{2.17}$$

Integrating by parts, we obtain that

(2)

$$\left\langle \prod_{\ell=1}^{n} \frac{1}{2\pi} ({}^*db)^{\mu_\ell}(x_\ell) \right\rangle$$

$$= \Xi^{-1} \int \mathcal{D}[b] \int \mathcal{D}[A] \int \mathcal{D}\Psi\, \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)}$$

$$\times \prod_{\ell=1}^{n} \left(-\mathrm{i} \frac{\delta}{\delta A_{\mu_\ell}(x_\ell)} \right) \exp\left(\frac{\mathrm{i}}{2\pi} \int A \wedge db \right)$$

$$= \Xi^{-1} \int \mathcal{D}[b]\, \mathcal{D}[A] \prod_{\ell=1}^{n} \left(\mathrm{i} \frac{\delta}{\delta A_{\mu}(x_\ell)} \right) \left(\int \mathcal{D}\Psi; \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)} \right)$$

$$\times \exp\left(\frac{\mathrm{i}}{2\pi} \int A \wedge db \right)$$

$$= \Xi^{-1} \int \mathcal{D}[b]\, \mathcal{D}[A] \int \mathcal{D}\Psi\, \mathcal{D}\Psi^*\, \mathrm{e}^{-I(\Psi,\Psi^*,A)} \prod_{\ell=1}^{n} \left(-\mathrm{i} \frac{\delta}{\delta A_{\mu}(x_\ell)} \right) I(\Psi, \Psi^*, A)$$

$$\times \exp\left(\frac{\mathrm{i}}{2\pi} \int A \wedge db \right)$$

$$= \Xi^{-1} \int \mathcal{D}[A] \int \mathcal{D}\Psi\, \mathcal{D}\Psi^* \mathrm{e}^{-I(\Psi,\Psi^*A)} \delta(dA) \prod_{\ell=1}^{n} \mathcal{J}^{\mu_\ell}(\Psi, \Psi^*, A; x_\ell) . \tag{2.18}$$

□

Our explicit calculation, equation (2.18), shows that identity (2.16) also holds at coinciding points if the current correlation functions in the fermion theory are defined by

$$\left\langle \prod_{\ell=1}^{n} \mathcal{J}^{\mu_\ell}(\Psi, \Psi^*; x_\ell) \right\rangle \equiv \int \mathcal{D}[A]\, \delta(dA) \prod_{\ell=1}^{n} \left(\mathrm{i} \frac{\delta}{\delta A_{\mu_\ell}(x_\ell)} \right) \int \mathcal{D}\Psi\, \mathcal{D}\Psi^* \mathrm{e}^{-I(\Psi, \Psi^*, A)} . \quad (2.19)$$

Remark 2.2. With the definition (2.19) the current correlation functions are transversal in the fermionic theory: by gauge invariance of $S(A)$, we have that for an arbitrary test function χ

$$\int \mathcal{D}[A]\, \delta(dA) \prod_{\ell=1}^{n-1} \left(-\mathrm{i} \frac{\delta}{\delta A_{\mu_\ell}(x_\ell)} \right) \mathrm{e}^{-S(A+\alpha d\chi)}$$
$$= \int \mathcal{D}[A]\, \delta(dA) \prod_{\ell=1}^{n-1} \left(-\mathrm{i} \frac{\delta}{\delta A_{\mu_\ell}(x_\ell)} \right) \mathrm{e}^{-S(A)} \quad (2.20)$$

and, differentiating (2.20) with respect to α and setting α to 0, we conclude that

$$\int d^{d+1}x_n\, \partial_{\mu_n} \chi(x_n) \int \mathcal{D}[A]\, \delta(dA) \prod_{\ell=1}^{n} \left(-\mathrm{i} \frac{\delta}{\delta A_{\mu_\ell}(x_\ell)} \right) \mathrm{e}^{-S(A)}$$
$$= \int d^{d+1}x_n\, \partial_{\mu_n} \chi(x_n) \langle \mathcal{J}^{\mu_1}(\Psi, \Psi^*; x_1) \cdots \mathcal{J}^{\mu_n}(\Psi, \Psi^*; x_n) \rangle = 0 . \quad (2.21)$$

Note that if $I(\Psi^*, \Psi, A)$ is linear in, e.g. the component A_0, then the definition of $\mathcal{J}^0$ through (2.19) coincides with the one given in (2.15).

Remark 2.3. Instead of using measures on gauge equivalence classes for A and b in (2.11), (2.14), one can add to the actions $S(A)$ and $\tilde{S}(db)$, for $d > 1$, the standard gauge fixing and Faddeev–Popov terms. For gauge forms of rank $k > 1$, this involves a tower of gauge fixings and ghosts, as explained in [11], for example.

In summary, equation (2.14) expresses the partition function of our theory, originally formulated in terms of the fermionic field Ψ, as the partition function of the bosonic gauge form b.

Equation (2.16) proves that the correlation functions of the physical current of the fermionic theory, $\mathcal{J}^\mu(\Psi, \Psi^*)$, are given by the correlation function of the dual of the curvature, or 'field strength', $({}^*db)^\mu$, of the gauge form b. This result has some interesting applications. We consider a system with an action $I_{\text{tot}}(\Psi, \Psi^*)$ given by a perturbation of $I(\Psi, \Psi^*)$ by a current–current interaction, in the form of a term, $I_{\text{pert}}(\Psi, \Psi^*)$, given by

$$I_{\text{pert}}(\Psi, \Psi^*) = \tfrac{1}{2} \int V_{\mu\nu}(x, y) \mathcal{J}^\mu(\Psi, \Psi^*; x) \mathcal{J}^\nu(\Psi, \Psi^*; y)$$
$$\equiv \tfrac{1}{2} \left(\mathcal{J}(\Psi, \Psi^*), V \mathcal{J}(\Psi, \Psi^*) \right) . \quad (2.22)$$

This perturbation, quartic in the Ψ field, becomes quadratic in the b field. In fact, adopting the definition (2.19), we obtain, for the partition function $\Xi(V)$ of the perturbed theory, the expression

$$\Xi(V) = \int \mathcal{D}\Psi\, \mathcal{D}\Psi^* \exp\left\{-\left[I(\Psi,\Psi^*) + \tfrac{1}{2}(\mathcal{J}(\Psi,\Psi^*), V\mathcal{J}(\Psi,\Psi^*))\right]\right\}$$

$$+ \int \mathcal{D}[b] \int \mathcal{D}[A] \int \mathcal{D}\Psi\, \mathcal{D}\Psi^* \left(\exp\left(\tfrac{1}{2}\left(\frac{\delta}{\delta A}, V\frac{\delta}{\delta A}\right)\right)\right.$$

$$\left.\times \exp\left(-I(\Psi,\Psi^*,A)\right)\right) \exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge db\right)$$

$$= \int \mathcal{D}[b] \int \mathcal{D}[A] \left(\exp\left(\tfrac{1}{2}\left(\frac{\delta}{\delta A}, V\frac{\delta}{\delta A}\right)\right)\right.$$

$$\left.\times \exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge db\right)\right) \mathrm{e}^{-S(A)}$$

$$= \int \mathcal{D}[b] \int \mathcal{D}[A]\, \exp\left(-\frac{1}{8\pi^2}({}^*db, V{}^*db)\right) \mathrm{e}^{-S(A)} \exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge db\right)$$

$$= \int \mathcal{D}[b]\, \exp\left\{-\left[\tilde{S}(db) + \frac{1}{8\pi^2}({}^*db, V{}^*db)\right]\right\}. \tag{2.23}$$

Clearly, these somewhat abstract identities become useful only if $\tilde{S}(db)$ has a tractable form. In particular, if $S(A)$ is quadratic, the perturbed theory has an action that is still quadratic in b. In later sections, we briefly discuss some systems where this appealing situation is encountered, i.e. where $S(A)$ is quadratic in A: massless relativistic fermions in one dimension, the 'scaling limit' of finite–density, non-relativistic free fermions in arbitrary dimensions, the 'scaling limit' of the bulk effective action of an incompressible quantum Hall fluid, or an insulator.

3. Disorder fields and fermion correlation functions

In section 2 we have reformulated a fermionic theory with global $U(1)$-gauge invariance as a gauge theory (for $d > 1$) of gauge forms of rank $d-1$.

Here we wish to discuss the question how correlation functions of the field Ψ and Ψ^* can be expressed in the bosonic theory. A general result is that they involve a disorder field conjugate to the gauge form b. We first propose a definition of disorder fields for gauge forms in general terms.

Let us denote by $\tilde{S}(db)$ the action of a gauge theory of $(d-1)$-forms in $M = \mathbb{R}^{d+1}$. We choose n points $\underline{x} = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^{d+1}$ and define

$$M_{\underline{x}} = \mathbb{R}^{d+1} \setminus \{x_1, \ldots, x_n\}. \tag{3.1}$$

Since $H^d_{\mathrm{deR}}(M_{\underline{x}}) \neq 0$ gauge forms $\tilde{b}$ whose curvature $f(\tilde{b})$ belongs to a non-trivial cohomology class in $H^d_{\mathrm{deR}}(M_{\underline{x}})$ are not globally defined on $M_{\underline{x}}$ (see the appendix).

We choose n non-zero integers, $\underline{q} = \{q_1, \ldots, q_n\}$, satisfying $\sum_{i=1}^n q_i = 0$ and define the closed d-form

$$\varphi_{\underline{x};\underline{q}} = \sum_{i=1}^{n} \varphi_{x_i,q_i} \qquad \text{where } \varphi_{x_i;q_i} = 2\pi q_i\, {}^*d\Delta^{-1}\delta_{x_i} \tag{3.2}$$

with $\delta_{x_i} \equiv \delta(x_i - z)$. One easily checks that, for every closed d-dimensional surface $S_{(d)}$ in $M_{\underline{x}}$,

$$\frac{1}{2\pi} \int_{S_{(d)}} \varphi_{\underline{x};\underline{q}} \in \mathbb{Z} . \tag{3.3}$$

More precisely, for a $(d+1)$-dimensional ball, B^i, in $\mathbb{R}^{d+1}$ with $x_i \in B^i$, $x_j \notin B^i$, for $j \neq i$, we have that

$$\frac{1}{2\pi} \int_{\partial B^i} \varphi_{\underline{x};\underline{q}} = q_i \tag{3.4}$$

where ∂ denotes the boundary. Thanks to equation (3.3), $\varphi_{\underline{x};\underline{q}}$ is the curvature of a gauge form $\tilde{\alpha}_{\underline{x};\underline{q}}$, of rank $d-1$ in $M_{\underline{x}}$. Note that

$$\delta\varphi_{\underline{x};\underline{q}} = 0 \tag{3.5}$$

and

$$d\varphi_{\underline{x};\underline{q}} = 2\pi \sum_{i=1}^{n} q_i \, {}^*\delta_{x_i} . \tag{3.6}$$

Equations (3.5) and (3.6) imply that $\varphi_{\underline{x};\underline{q}}$ is a harmonic form in $M_{\underline{x}}$.

In physics terminology, $\varphi_{x;q}$ is the vector potential of a magnetic vortex at the point x when d=1, while, for d=2, it is the magnetic field of a monopole located at x, with magnetic charge q. The curvature of the gauge forms $\tilde{b}$ on $M_{\underline{x}}$ can be written as $f(\tilde{b}) = db + \varphi_{\underline{x},\underline{y}}$.

The expectation value of a disorder field

$$D(\underline{x}, \underline{q}) \equiv \prod_{i=1}^{n} D(x_i, q_i) \tag{3.7}$$

is given by

$$\langle D(\underline{x}, \underline{q})\rangle = \left\{ \Xi^{-1} \int \mathcal{D}[b] \, \exp\left(-\tilde{S}(db + \varphi_{\underline{x};\underline{q}})\right) \right\}_{\text{ren}} \tag{3.8}$$

where

$$\Xi = \int \mathcal{D}[b] \, \mathrm{e}^{-\tilde{S}(db)}$$

is the partition function [12, 13].

On the r.h.s. of (3.8) a multiplicative renormalization is usually necessary. This has been discussed for three-dimensional Higgs theories in [10]. We shall in general omit the subscript 'ren'. Formally, equation (3.8) can be interpreted as the expectation value of

$$D(\underline{x}, \underline{q}) = \exp\left\{ - \left[\tilde{S}(db + \varphi_{\underline{x};\underline{q}}) - \tilde{S}(db) \right] \right\} . \tag{3.9}$$

Correlation functions involving $D(\underline{x}, \underline{q})$ and gauge–invariant functionals of db and b, $\mathcal{F}(db, b)$, are defined by

$$\left\langle D(\underline{x}, \underline{q}) \mathcal{F}(db, b) \right\rangle = \Xi^{-1} \int \mathcal{D}[b] \, \mathcal{F}(db + \varphi_{\underline{x};\underline{q}}, b + \tilde{\alpha}_{\underline{x};\underline{q}}) \exp\left(-\tilde{S}(db + \varphi_{\underline{x};\underline{q}})\right) \tag{3.10}$$

with $\varphi_{\underline{x};\underline{q}} = d\tilde{\alpha}_{\underline{x};\underline{q}}$.

Equation (3.10) can be understood by introducing an action from the left of $D(x_i, q_i)$ on $\mathcal{F}(db, b)$, namely

$$D(x_i, q_i)\mathcal{F}(db, b) \equiv \mathcal{F}(db + \varphi_{x_i;q_i}, b + \tilde{\alpha}_{x_i;q_i})D(x_i, q_i)\,. \tag{3.11}$$

It is well known that from the Euclidean correlation functions of $db(x)$ (satisfying a suitable variant of the Osterwalder–Schrader (OS) axioms) [14] one can reconstruct the vacuum state $|0\rangle$, the Hilbert space $\mathcal{H}$, the Hamiltonian H and field operators $\hat{j}(x) = {}^*\widehat{db}(x)$ of a quantum field theory. The relation between the Euclidean field db and the field operator $\widehat{db}$ is given as follows. Let $x_1^0 < x_2^0 < \cdots < x_n^0$, and define

$$\hat{O}(x) \equiv \mathrm{e}^{-x^0 H}\hat{O}(x)\mathrm{e}^{x^0 H}\,. \tag{3.12}$$

Then

$$\langle db(x_1)\cdots db(x_n)\rangle = \Xi^{-1}\int \mathcal{D}[b]\,\mathrm{e}^{-\bar{S}(db)}db(x_1)\cdots db(x_n) = \langle 0|\widehat{db}(x_1)\cdots\widehat{db}(x_n)|0\rangle. \tag{3.13}$$

Extending this result, it has been proved in [13], that from the correlation functions of the disorder fields (satisfying a suitable version of the OS axioms) one can reconstruct soliton field operators $\hat{S}_q(x)$ such that, for $x_1^0 < \cdots < x_n^0$,

$$\langle D(\underline{x}, \underline{y})\rangle = \langle 0|\hat{S}_{q_1}(x_1)\cdots\hat{S}_{q_n}(x_n)|0\rangle\,. \tag{3.14}$$

*Similar identies hold for mixed correlation functions involving disorder fields and gauge-*invariant functionals of db and b.

From equations (3.12) and (3.13) it follows that one may consider $D(x, q)$ as the Euclidean field corresponding to the soliton field operator $\hat{S}_q(x)$ at Euclidean time x^0.

Given a gauge form $\tilde{b}^{(d-1)}$ and a d-dimensional surface $\Sigma_{(d)}$, one defines the 'Wilson operator of rank $(d-1)$' by

$$W_\alpha(\Sigma_{(d)}) = \left\{\exp\left(\mathrm{i}\alpha\int_{\Sigma_{(d)}} f(\tilde{b}^{(d-1)})\right)\right\}_{\mathrm{ren}} \tag{3.15}$$

where α is a real number. We observe that if $\tilde{b}^{(d-1)}$ is a globally defined form, denoted by b, then

$$W_\alpha(\Sigma_{(d)}) = \left\{\exp\left(\mathrm{i}\alpha\int_{\mathcal{L}_{(d-1)}} b\right)\right\}_{\mathrm{ren}} \tag{3.16}$$

where $\mathcal{L}_{(d-1)}$ is the boundary, $\partial\Sigma_{(d)}$, of $\Sigma_{(d)}$, and $W_\alpha(\Sigma_{(d)})$ coincides with the ordinary Wilson loop when $d = 2$. On the r.h.s. of (3.15), (3.16) a multiplicative renormalization, depending on α and $\mathcal{L}_{(d-1)}$ is usually necessary, but we shall in general omit the subscript 'ren'. The renormalization can be made precise for theories with a quadratic action [15].

If $\Sigma_{(d)}$ is contained in a fixed-time (hyper-) plane we shall write $W_\alpha(\mathcal{L}_{(d-1)})$, instead of $W_\alpha(\Sigma_{(d)})$, and we denote by $\hat{W}_\alpha(\mathcal{L}_{(d-1)})$ the corresponding field operator. The soliton operator and the Wilson operator $\hat{W}_\alpha(\mathcal{L}_{(d-1)})$ satisfy the 'dual algebra' [5, 15]

$$\hat{S}_q(x)\hat{W}_\alpha(\mathcal{L}_{(d-1)}) = \begin{cases} \mathrm{e}^{\mathrm{i}2\pi\alpha q}\hat{W}(\mathcal{L}_{(d-1)})\hat{S}_q(x) & \text{for } x \in \operatorname{int}\Sigma_{(d)} \\ \hat{W}_\alpha(\mathcal{L}_{(d-1)})\hat{S}_q(x) & \text{for } x \notin \operatorname{int}\Sigma_{(d)}. \end{cases} \tag{3.17}$$

By T_ϵ we denote translation by ϵ in the positive time direction, and let $x = (0, \boldsymbol{x})$. Then the dual algebra implies the following equation:

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\Big\langle \cdots D(\underline{x},\underline{q})\cdots W_{-\alpha}(T_{-\epsilon}\mathcal{L}_{(d-1)})W_\alpha(T_\epsilon\mathcal{L}_{(d-1)})\cdots\Big\rangle \\
&= \lim_{\epsilon\downarrow 0}\Bigg\langle \cdots \exp\left(-\mathrm{i}\alpha\int_{T_{-\epsilon}\Sigma_{(d)}}(db+\varphi_{\underline{x};\underline{q}})\right) \\
&\quad\times \exp\left(\mathrm{i}\alpha\int_{T_\epsilon\Sigma_{(d)}}(db+\varphi_{\underline{x};\underline{q}})\right) D(\underline{x},\underline{q})\cdots\Bigg\rangle \\
&= \lim_{\epsilon\downarrow 0}\exp\left(\mathrm{i}\alpha\left(\int_{T_\epsilon\Sigma_{(d)}}\varphi_{\underline{x};\underline{q}} - \int_{T_{-\epsilon}\Sigma_{(d)}}\varphi_{\underline{x};\underline{q}}\right)\right) \\
&\quad\times\left\langle\cdots\exp\left(-\mathrm{i}\alpha\int_{T_{-\epsilon}\Sigma_{(d)}}db+\mathrm{i}\alpha\int_{T_\epsilon\Sigma_{(d)}}db\right)D(\underline{x},\underline{q})\cdots\right\rangle \\
&= \exp\left(\mathrm{i}\alpha\sum_{j:\boldsymbol{x}_j\in\mathrm{int}\,\Sigma_{(d)}} q_j\right)\Big\langle\cdots D(\underline{x},\underline{q})\cdots\Big\rangle\,. \qquad (3.18)
\end{aligned}$$

Let $\hat{\Psi}_\alpha$ and $\hat{\mathcal{J}}^\mu$ denote the fermion field operator and the current operator reconstructed from the correlation functions of $\Psi_\alpha, \Psi^*_\alpha$ and $\mathcal{J}^\mu(\Psi,\Psi^*)$ in the fermionic theory and let $\hat{\mathcal{Q}}_{\Sigma_{(d)}}$ denote the $U(1)$ charge operator associated with a d-dimensional surface, $\Sigma_{(d)}$, contained in the time-zero plane; formally

$$\hat{\mathcal{Q}}_{\Sigma_{(d)}} = \int_{\Sigma_{(d)}} d^d x\; \hat{\mathcal{J}}^0(\boldsymbol{x})\,. \qquad (3.19)$$

The fact that $\hat{\Psi}^\#_\beta(\boldsymbol{x}) = \hat{\Psi}_\beta(\boldsymbol{x}), \left(\hat{\Psi}^\dagger_\beta(\boldsymbol{x})\right)$ carries charge $+1$, (-1) localized at $\boldsymbol{x}$ is summarized in the equation

$$\begin{aligned}
\langle 0|T\left(\cdots\exp\left(-\mathrm{i}\alpha\hat{\mathcal{Q}}_{\Sigma_{(d)}}\right)\hat{\Psi}^\#_\beta(\boldsymbol{x})\exp\left(\mathrm{i}\alpha\hat{\mathcal{Q}}_{\Sigma_{(d)}}\right)\cdots\right)|0\rangle \\
&= \lim_{\epsilon\downarrow 0}\langle 0|T\left(\cdots\exp\left(-\mathrm{i}\alpha\hat{\mathcal{Q}}_{T_{-\epsilon}\Sigma_{(d)}}\hat{\Psi}^\#_\beta(\boldsymbol{x})\right)\exp\left(\mathrm{i}\alpha\hat{\mathcal{Q}}_{T_\epsilon\Sigma_{(d)}}\right)\cdots\right)|0\rangle \\
&= \lim_{\epsilon\downarrow 0}\Bigg\langle T\Bigg(\cdots\exp\left(-\mathrm{i}\alpha\int_{T_{-\epsilon}\Sigma_{(d)}}d^d x\;\mathcal{J}^0(x)\right)\Psi^\#_\beta(0,\boldsymbol{x}) \\
&\quad\times\exp\left(\mathrm{i}\alpha\int_{T_\epsilon\Sigma_{(d)}}d^d x\;\mathcal{J}^0(x)\right)\cdots\Bigg)\Bigg\rangle \\
&= \Big\langle T\left(\cdots\Psi^\#_\beta(0,\boldsymbol{x})\cdots\right)\Big\rangle\begin{cases}\mathrm{e}^{(\pm)\mathrm{i}\alpha} & \boldsymbol{x}\in\mathrm{int}\;\Sigma_{(d)}\\ 1 & \boldsymbol{x}\notin\Sigma_{(d)}\end{cases} \\
&= \langle 0|T\left(\cdots\hat{\Psi}^\#_\beta(\boldsymbol{x})\cdots\right)|0\rangle\begin{cases}\mathrm{e}^{(\pm)\mathrm{i}\alpha} & \boldsymbol{x}\in\mathrm{int}\;\Sigma_{(d)}\\ 1 & \boldsymbol{x}\notin\Sigma_{(d)}\end{cases} \qquad (3.20)
\end{aligned}$$

where $T(\cdot)$ denotes Euclidean time ordering.

We recall that $\mathcal{J}^0(\Psi, \Psi^*)$ is represented in the bosonic theory by $\frac{1}{2\pi}({}^*db)^0$. Comparison of (3.18) and (3.20) then shows that the insertion of $\Psi_\alpha(x)$, $\Psi^*_\alpha(x)$, in the fermionic theory implies the insertion of the disorder field $D(x, 1)$, $D(x, -1)$ in the bosonized theory. In many examples we have a more explicit relation, namely

$$\Psi_\alpha(x) = D(x, 1)\mathcal{F}_\alpha(x; b) \qquad \Psi^*_\alpha(x) = D(x, -1)\mathcal{F}^*_\alpha(x; b)$$

where $\mathcal{F}_\alpha(x; b)$ is a functional of b that has been determined in $(1+1)$-dimensional theories (see section 4) and in some $(2+1)$-dimensional Chern–Simons theories [17].

Hence, fermion fields in the bosonic theory are 'proportional' to disorder fields, and fermions can be viewed as solitons of the bosonic theory.

There is a class of models where the relation between disorder fields and fermion fields can be made more explicit. Consider a system of fermions with an action $I(\Psi, \Psi^*)$, perturbed by a current–current interaction (2.22). We denote by $\langle(\cdot)\rangle^V$ the Euclidean expectation value of the perturbed system and by $\langle(\cdot)\rangle_A$ the expectation value corresponding to the unperturbed gauge-invariant action $I(\Psi, \Psi^*, A)$. Suppose that the fermion correlation functions are expressible through a formula

$$\left\langle T\left(\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}(x_n)\Psi_{\alpha_{n+1}}(x_{n+1})\cdots\Psi_{\alpha_{2n}}(x_{2n})\right)\right\rangle_A = \int d\mu_{\underline{x};\underline{\alpha}}(J^{(1)})\,\mathrm{e}^{\mathrm{i}(A,J)} \tag{3.21}$$

where $d\mu_{\underline{x};\underline{\alpha}}(J^{(1)})$ is a measure on the space of 1-forms $J^{(1)}$. The transformation property under $U(1)$-gauge transformations of the fermion fields, implies that $d\mu(J^{(1)})$ is supported on forms satisfying

$$\delta J^{(1)}(z) = \sum_{i=1}^{n}\delta_{x_i}(z) - \sum_{i=n+1}^{2n}\delta_{x_i}(z)\,.$$

Let $q_i = 1$ for $i = 1, \ldots, n$, $q_i = -1$ for $i = n+1, \ldots, 2n$. Then, with $\varphi_{\underline{x};\underline{q}}$ as in (3.2),

$$\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}(x_n)\Psi_{\alpha_{n+1}}(x_{n+1}))\cdots\Psi_{\alpha_{2n}}(x_{2n})\exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge\varphi_{\underline{x};\underline{q}}\right) \tag{3.22}$$

is gauge-invariant. Following (2.23) we derive the identity

$$\begin{aligned}
\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi_{\alpha_{2n}}(x_{2n}))\rangle^V &= \Xi(V)^{-1}\int\mathcal{D}[b]\int\mathcal{D}[A]\,\mathcal{D}\Psi\,\mathcal{D}\Psi^*\,T\big(\Psi^*_{\alpha_1}(x_1)\\
&\quad\cdots\Psi_{\alpha_{2n}}(x_{2n})\big)\exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge\varphi_{\underline{x};\underline{q}}\right)\\
&\quad\times\exp\left(-I(\Psi,\Psi^*,A) - \tfrac{1}{2}(\mathcal{J}(\Psi,\Psi^*,A), V\mathcal{J}(\Psi,\Psi^*,A))\right)\\
&\quad\times\exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge db\right)\\
&= \Xi(V)^{-1}\int\mathcal{D}[b]\left(\int\mathcal{D}[A]\,\mathrm{e}^{-S(A)}\big\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi_{\alpha_{2n}}(x_{2n}))\big\rangle_A\right.\\
&\quad\left.\times\exp\left(\frac{\mathrm{i}}{2\pi}A\wedge(db+\varphi_{\underline{x};\underline{q}})\right)\exp\left(-\frac{1}{8\pi^2}\left({}^*(db+\varphi_{\underline{x};\underline{q}}), V^*(db+\varphi_{\underline{x};\underline{q}})\right)\right)\right).
\end{aligned} \tag{3.23}$$

Inserting (3.21) and integrating over A, we obtain that

$$\begin{aligned}\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi_{\alpha_{2n}}(x_{2n}))\rangle^V \\ &= \Xi(V)^{-1}\int \mathcal{D}[b]\int d\mu_{\underline{x};\underline{\alpha}}(J)\,\exp\left(-\tilde{S}(db+\varphi_{\underline{x};\underline{q}}+2\pi^* J)\right) \\ &\quad\times\exp\left(-\frac{1}{8\pi^2}\left({}^*(db+\varphi_{\underline{x};\underline{q}}),V^*(db+\varphi_{\underline{x},\underline{q}})\right)\right).\end{aligned} \tag{3.24}$$

The r.h.s. of (3.24) has the desired form of an expectation value of a disorder field.

Remark 4.2. The representation in (3.21) of correlation functions of fermions, holds, e.g., for a system of free non-relativistic fermions with chemical potential μ, at temperature T. The measure in (3.21) is defined as follows: let $\underline{\omega} = \{\omega_1,\ldots,\omega_n\}$ denotes a set of (Brownian) paths joining $\underline{x} = \{x_1,\ldots,x_n\}$ to $\underline{y} = \{x_{n+1},\ldots,x_{2n}\}$, and let $\underline{\alpha} = \{\alpha_1,\ldots,\alpha_n\}, \underline{\delta} = \{\alpha_{n+1},\ldots,\alpha_{2n}\}$. Formally we set

$$J_{\underline{\omega}}(z) = \sum_{\ell=1}^{n}\left(dz^0\delta\big(z-\omega_\ell(z^0)\big) + dz^i\frac{dz_i}{dz^0}\delta\big(z-\omega_\ell(z^0)\big)\right). \tag{3.25}$$

Then

$$\begin{aligned}\int d\mu(J_{\underline{\omega}})\,(\cdot) &= \sum_{\pi\in\Sigma_n}(-1)^{\sigma(\pi)}\prod_{k=1}^{n}\left(\sum_{\ell_k=0,1,\ldots}(-1)^{\ell_k}\Theta(y^0_{\pi(k)}+\ell_k\beta-x^0_k)\right) \\ &\quad\times\int_{\substack{\omega(x^0_k)=x_k \\ \omega(y^0_{\pi(k)}+\ell_k\beta)=y_{\pi}(k)}}\prod_{k=1}^{n}\Bigg[\mathcal{D}\omega_k\exp\left\{-\int_{x^0_k}^{y^0_{\pi(k)}+\ell_k\beta}dt\,\frac{m}{2}\dot{\omega}^2_k(t)\right\} \\ &\quad\times\exp\left(\mu(y^0_{\pi(k)}+\ell_k\beta-x^0_k)\right)\delta_{\alpha_k,\delta_{\pi(k)}}\Bigg](\cdot)\end{aligned} \tag{3.26}$$

where Σ_n is the group of permutations of n objects, $\sigma(\pi)$ is the signature of $\pi\in\Sigma_n$, Θ denotes the Heaviside step function, m the mass of the particles and $\beta = \frac{1}{kT}$, where k is the Boltzmann constant. For a derivation of this formula see [16]. In [17] the factor $(-1)^{\ell_k}$ was erroneously missing. A similar formula for relativistic, massive, spin-$\frac{1}{2}$ fermions can be found in [18, 19].

To conclude this section we recall an interesting relation between Wilson loops and disorder fields, namely the 't Hooft duality [5]. Let $\Sigma^R_{(d)}$ denote a d-dimensional ball of radius R in the Euclidean time-zero plane, let $\mathcal{L}^R_{(d-1)} \equiv \partial\Sigma^R_{(d)}$ and let $W_\alpha(\mathcal{L}^R_{(d-1)})$ be the corresponding Wilson loop of rank $d-1$. Then the *'t Hooft duality* is the following conjecture, verified in many concrete models. If, for any $c>0$, $\alpha\neq 0$,

$$\exp\left(c|\mathcal{L}^R_{(d-1)}|\right)\langle W_\alpha(\Sigma^R_{(d)})\rangle \to 0 \qquad \text{as } R\to\infty \tag{3.27}$$

where $|\mathcal{L}|$ denotes the volume of $\mathcal{L}$, then the expectation value $\langle D(x,1;y,-1)\rangle$ of the disorder field $D(x,1;y,-1)$, (with x,y in the time-zero plane) has decay slower than exponential in $(x-y)$. If, for some $c<\infty$,

$$\langle W_\alpha(\mathcal{L}^R_{(d-1)})\rangle \geqslant \exp\left(-c|\mathcal{L}^R_{(d-1)}|\right) \qquad \text{as } R\to\infty \tag{3.28}$$

i.e. the Wilson loop has perimeter decay or slower, then either $\langle D(x,1;y-1)\rangle$ has exponential decay in $(x-y)$, or the density, ${}^*(db)^0$, correlation functions have a gapless mode.

There is also a criterion for the existence of a global $U(1)$-charge. In the Euclidean formalism the charge operator $\hat{Q}$ for the bosonic theory is defined by the following (weak) limit:

$$e^{i\alpha 2\pi\hat{Q}} = \lim_{R\to\infty} \frac{\hat{W}_\alpha(\mathcal{L}^R_{(d-1)})}{\langle 0|\hat{W}_\alpha(\mathcal{L}^R_{(d-1)})|0\rangle} \,. \tag{3.29}$$

The denominator is needed in order to ensure that the charge (if it exists) annihilates the vacuum $|0\rangle$. A heuristic criterion for the existence of the limit in (3.29) is that $\langle 0|\hat{W}_\alpha(\mathcal{L}^R_{(d-1)})|0\rangle = \langle W_\alpha(\mathcal{L}^R_{(d-1)})\rangle$ have, at most, perimeter decay. This criterion has been proved to be correct for many lattice gauge theories [20]. We propose to extend this criterion, as well as the 't Hooft duality, to systems of condensed matter physics in the bosonized representation developed in this and the previous section.

4. Relativistic massless fermions in $d = 1$

In this section we show that the construction outlined in sections 2 and 3 reduces to ordinary Abelian bosonization [21] if $I(\Psi, \Psi^*)$ is the action of massless free Dirac fermions in d=1. Although we do not pretend to have any new results, we feel it is useful to illustrate the general theory described in sections 2 and 3 in this simple situation. In fact, our discussion sheds some new light on the principles underlying bosonization in two dimensions.

We start by recalling the main formulae of Abelian bosonization. Let $\gamma^i, i = 0, 1, 5$ be two-dimensional Euclidean Dirac matrices, identified, in the chiral basis, with the Pauli matrices $\sigma_2, \sigma_1, \sigma_3$. The Euclidean Dirac operator is given by

$$\partial\!\!\!/ = \gamma^0\partial_0 + \gamma^1\partial_1$$

and the Euclidean action of the massless Dirac field is given by

$$I(\Psi, \Psi^*) = \int d^2x\ \Psi^*_\alpha \partial\!\!\!/ \Psi_\alpha(x) \tag{4.1}$$

where $\Psi_\alpha, \Psi^*_\alpha = 1, 2$ are two-component Grassmann fields.

The Euclidean action for the massless, relativistic, real scalar field, Φ, is given by

$$\tilde{S}(d\Phi) = \frac{1}{8\pi}\int d^2x\ (\partial_\mu\Phi)^2(x) = \frac{1}{8\pi}(d\Phi, d\Phi)\,. \tag{4.2}$$

The simplest bosonization formulae, proved by direct computation, are the following: the Euclidean correlation functions of $\mathcal{J}^\mu(\Psi, \Psi^*) = :\Psi^*\gamma^\mu\Psi:$ and $:\Psi^*\frac{(1\pm\gamma_5)}{2}\Psi:$,normal ordered in the fermionic theory, are identical to the Euclidean correlations of $\frac{1}{2\pi}\epsilon_{\mu\nu}\partial^\nu\Phi$ and $:e^{\pm i\Phi}:$, normal ordered in ths bosonic theory. The normal ordering chosen in the bosonic theory can be characterized formally by

$$:e^{i\alpha\Phi(x)} := e^{i\alpha\Phi(x)}(2\pi)^{\alpha^2}\exp\left(2\pi\alpha^2\Delta^{-1}(x,x)\right) \tag{4.3}$$

where $\Delta^{-1}(x, y) \equiv \frac{1}{2\pi} \ln |x - y|$. (For a rigorous definition one uses a point-splitting regularization.)

One can also give explicit bosonization formulae for the fields Ψ and Ψ^*. The formulae for the above bilinear expressions then follow by taking limits [22]. We briefly recall some details of the result.

Let $\underline{x} = \{x_1, \dots, x_n\}$, $\underline{y} = \{y_1, \dots, y_n\}$, $\underline{q} = \{q_i\}_{i=1}^n$, $\underline{q}' = \{q_j'\}_{j=1}^n$, $q_i = -1$, $q_j' = +1$. One first constructs the corresponding closed 1-form (4.2), $\varphi_{\underline{x},\underline{y};\underline{q},\underline{q}'} \equiv \varphi_{\underline{x},\underline{y}}$ and the corresponding rank-0 gauge form $\tilde{\alpha}_{\underline{x},\underline{y};\underline{q},\underline{q}'} \equiv \tilde{\alpha}_{\underline{x},\underline{y}}$. One can give an explicit expression for $\tilde{\alpha}_{\underline{x},\underline{y}}$ as the multivalued function

$$\tilde{\alpha}_{\underline{x},\underline{y}}(z) = \sum_{i=1}^{n} (\arg(x_i - z) - \arg(y_i - z)) \tag{4.4}$$

where arg $x, x \in \mathbb{R}^2$ denotes the argument of $\mathrm{i}x^0 + x^1$.

According to definition (3.8) the expectation value of the disorder field is given by

$$\langle D(\underline{x}, \underline{q}, \underline{y}, \underline{q}')\rangle \equiv \langle D(\underline{x}, \underline{y})\rangle = \left\{ \frac{\int \mathcal{D}\Phi \ \exp\left(-\tilde{S}(d\Phi + \varphi_{\underline{x},\underline{y}})\right)}{\int \mathcal{D}\Phi \ \mathrm{e}^{-\tilde{S}(d\Phi)}} \right\}_{\mathrm{ren}} . \tag{4.5}$$

To give the explicit form of the renormalization, notice that the term quadratic in $\varphi_{\underline{x},\underline{y}}$ in $\tilde{S}(d\Phi + \varphi_{\underline{x},\underline{y}})$ is logarithmically divergent: formally

$$\frac{1}{8\pi}(\varphi_{\underline{x},\underline{y}}, \varphi_{\underline{x},\underline{y}}) = \frac{1}{8\pi} \sum_{i,j=1}^{n} \left\{ 2q_i q_j' \Delta^{-1}(x_i, y_j) + q_i q_j \Delta^{-1}(x_i, x_j) + q_i' q_j' \Delta^{-1}(y_i, y_j) \right\} . \tag{4.6}$$

A multiplicative renormalization is then necessary, as mentioned after equation (3.8), to eliminate the terms with coinciding points in (4.6). This can be done as follows: let $S_\delta(x)$ denote a small ball of radius δ around x, and set

$$\begin{aligned} c(\delta) &\equiv \frac{1}{8\pi} \left(\sum_{i=1}^{n} \int_{\partial S_\delta(x_i)} + \sum_{j=i}^{n} \int_{\partial S_\delta(y_j)} \right) \varphi_{\underline{x},\underline{y}} \wedge {}^* d\Delta^{-1} \varphi_{\underline{x},\underline{y}} \\ &\underset{\delta \to 0}{\longrightarrow} \frac{\pi}{2} \left(\sum_{i=1}^{n} \Delta^{-1}(x_i, x_i) + \sum_{j=1}^{n} \Delta^{-1}(y_j, y_j) \right) . \end{aligned} \tag{4.7}$$

We define a regularized action by

$$\tilde{S}^\delta(d\Phi + \varphi_{\underline{x},\underline{y}}) = \frac{1}{8\pi} \int_{\mathbb{R}^2 \setminus S_\delta(\underline{x},\underline{y})} (d\Phi + \varphi_{\underline{x},\underline{y}}) \wedge {}^*(d\Phi + \varphi_{\underline{x},\underline{y}}) - c(\delta) \tag{4.8}$$

where $S_\delta(\underline{x}, \underline{y}) = \cup_{i=1}^n S_\delta(x_i) \cup_{j=1}^n S_\delta(y_j)$.

The precise definition of disorder field is given by

$$\langle D(\underline{x}, \underline{y})\rangle = \lim_{\delta \downarrow 0} \frac{\int \mathcal{D}\Phi \ \exp\left(-\tilde{S}^\delta(d\Phi + \varphi_{\underline{x},\underline{y}})\right)}{\int \mathcal{D}\Phi \mathrm{e}^{-\tilde{S}(d\Phi)}} . \tag{4.9}$$

We note here that from (4.7) it follows that the r.h.s. of (4.9) can be written as the r.h.s. of (4.5) with a multiplicative (infinite) renormalization

$$\exp\left(\frac{\pi}{2}\left(\sum_{i=1}^{n}\Delta^{-1}(x_i,x_1)+\sum_{j=1}^{n}\Delta^{-1}(y_j,y_j)\right)\right). \tag{4.10}$$

With the disorder field defined as above, one can prove the bosonization identity

$$\begin{aligned}\langle T\left(\Psi^*_{\alpha_1}(x_1)\cdots\Psi^*_{\alpha_n}\Psi_{\delta_1}(y_1)\cdots\Psi_{\delta_n}(y_n)\right)\rangle \\ = (2\pi)^{-\frac{n}{2}}\bigg\langle D(\underline{x},\underline{y})\prod_{i=1}^{n}:\exp\left(-\frac{\mathrm{i}}{2}(-1)^{\alpha_i}\Phi(x_i)\right): \\ \times\prod_{j=1}^{n}:\exp\left(-\frac{\mathrm{i}}{2}(-1)^{\delta_j}\Phi(y_j)\right):\bigg\rangle.\end{aligned} \tag{4.11}$$

Adding to $I(\Psi,\Psi^*)$ a current–current interaction (2.22), with $\mathcal{J}(\Psi,\Psi^*) =: \Psi^*\gamma_\mu\Psi:$, is equivalent to adding the term $\frac{1}{4\pi^2}({}^*d\Phi, V{}^*d\Phi)$ to $\tilde{S}(d\Phi)$ in the bosonic theory. If we denote by $\langle(\cdot)\rangle^V$ the corresponding expectation value, equation (4.11) holds for the perturbed theory replacing $\langle(\cdot)\rangle$ by $\langle(\cdot)\rangle^V$.

With the convention of writing the disorder field to the left of all functionals of Φ, the bosonization formula (4.11) yields the identifications

$$\begin{aligned}\Psi_1(x)&\to(2\pi)^{-\frac{1}{4}}D(x,1):\mathrm{e}^{\frac{\mathrm{i}}{2}\Phi(x)}:\\ \Psi_2(x)&\to(2\pi)^{-\frac{1}{4}}D(x,1):\mathrm{e}^{-\frac{\mathrm{i}}{2}\Phi(x)}:\\ \Psi_1^*(x)&\to(2\pi)^{-\frac{1}{4}}D(x,-1):\mathrm{e}^{+\frac{\mathrm{i}}{2}\Phi(x)}:\\ \Psi_2^*(x)&\to(2\pi)^{-\frac{1}{4}}D(x,-1):\mathrm{e}^{-\frac{\mathrm{i}}{2}\Phi(x)}:.\end{aligned} \tag{4.12}$$

We show how to derive the bosonization formula for $\langle T(\Psi^*_{\alpha_1}(x_1)\cdots\Psi_{\delta_n}(y_n))\rangle^V$, following the method outlined in sections 2 and 3. (Bilinear expressions in the fermion fields are bosonized by the same method, in [23]). We couple Ψ minimally to A, obtaining the gauge-invariant action

$$I(\Psi,\Psi^*,A)=\int d^2x\,\{\Psi^*(\partial\!\!\!/-A\!\!\!/)\Psi\}(x). \tag{4.13}$$

A standard computation, due originally to Schwinger, gives

$$\mathrm{e}^{-S(A)}=\exp\left(-\frac{1}{2\pi}(dA,\Delta^{-1}dA)\right)=\exp\left(-\frac{1}{2\pi}(A^{\mathrm{T}},A^{\mathrm{T}})\right). \tag{4.14}$$

In (4.14) A^{T} denotes the *transverse component* of A

$$A^{\mathrm{T}}\equiv\delta\Delta^{-1}dA=A-d\Delta^{-1}\delta A \tag{4.15}$$

where the second equality follows from the Hodge decomposition (in a space with $H^1_{\mathrm{deR}}(M) = 0$). In d=1 the field $b^{(0)}$ is just a scalar real field denoted by Φ. From equations (2.11)–(2.14) and (4.14) it follows that

$$e^{-\tilde{S}(d\Phi)} = \int \mathcal{D}[A]\, \exp\left(-\frac{1}{2\pi}(A^{\mathrm{T}}, A^{\mathrm{T}})\right) \exp\left(\frac{1}{2\pi}\int A \wedge d\Phi\right) = \exp\left(-\frac{1}{8\pi}(d\Phi, d\Phi)\right). \tag{4.16}$$

Hence, the action of $b^{(0)} \equiv \Phi$ coincides with (4.2). Next, we consider the fermion correlation functions. We note that, at non-coinciding points, the contributions coming from the left movers $\Psi_{\mathrm{L}} \equiv \Psi_1, \Psi_{\mathrm{L}}^* \equiv \Psi_2^*$ and the right movers $\Psi_{\mathrm{R}} \equiv \Psi_2, \Psi_{\mathrm{R}}^* \equiv \Psi_1^*$ factorize, so that one can simply consider correlation functions of right movers. Correlation functions of left movers are obtained by complex conjugation. The explicit expression of the correlation functions of free fermions in d=1 in the presence of a gauge field A is given by the equation (see [24])

$$\begin{aligned}\left\langle T\left(\Psi_{\mathrm{R}}^*(x_1)\cdots\Psi_{\mathrm{R}}^*(x_n)\Psi_{\mathrm{R}}(y_1)\cdots\Psi_{\mathrm{R}}(y_n)\right)\right\rangle_A \\ = \det\left(\frac{1}{x_i - y_j}\right)\exp\left\{\int d^2z\ \left[\mathrm{i}\delta\Delta^{-1}A(z) - {}^*d\Delta^{-1}A(z)\right]\right. \\ \left.\times\left[\sum_{i=1}^{n}\delta(z-x_i) - \sum_{j=1}^{n}\delta(z-y_j)\right]\right\}.\end{aligned} \tag{4.17}$$

We now combine equation (3.23) with (4.14) and (4.17) and, recalling equation (3.6), we obtain that

$$\begin{aligned}\left\langle T\left(\Psi_{\mathrm{R}}^*(x_1)\cdots\Psi_{\mathrm{R}}(y_n)\right)\right\rangle^V \\ = \Xi(V)^{-1}\int \mathcal{D}\Phi \int \mathcal{D}[A]\ \exp\left(-\frac{1}{2\pi}(A^{\mathrm{T}}, A^{\mathrm{T}})\right)\det\left(\frac{1}{x_i - y_j}\right) \\ \times\exp\left(\frac{\mathrm{i}}{2\pi}\int(\delta\Delta^{-1}A\wedge d\varphi_{\underline{x},\underline{y}} - A\wedge\varphi_{\underline{x},\underline{y}})\right)\exp\left(-\frac{1}{2\pi}(d\Delta^{-1}A, d\varphi_{\underline{x},\underline{y}})\right) \\ \times\exp\left(\frac{\mathrm{i}}{2\pi}\int A\wedge d\Phi\right)\exp\left(-\frac{1}{8\pi^2}\left({}^*(d\Phi + \varphi_{\underline{x},\underline{y}}), V^*(d\Phi + \varphi_{\underline{x},\underline{y}})\right)\right).\end{aligned} \tag{4.18}$$

Integrating out A, using the explicit form of $\varphi_{\underline{x},\underline{y}}$, equation (3.2), and the Cauchy identity

$$\begin{aligned}\det\left(\frac{1}{x_i - y_j}\right) &= \frac{\prod_{1\leqslant i<i'\leqslant n}(x_i - x_{i'})\prod_{1\leqslant j<j'\leqslant n}(y_{j'} - y_j)}{\prod_{1\leqslant i,j\leqslant n}(x_i - y_j)} \\ &= \frac{\prod_{1\leqslant i<i'\leqslant n}|x_i - x_{i'}|\prod_{1\leqslant j<j'\leqslant n}|y_j - y_{j'}|}{\prod_{1\leqslant i,j\leqslant n}|x_i - y_j|} \\ &\quad\times\prod_{1\leqslant i<i'\leqslant n}\exp\Big(\mathrm{i}\arg(x_i - x_{i'})\Big)\prod_{1\leqslant j\leqslant j'\leqslant n}\exp\Big(\mathrm{i}\arg(y_{j'} - y_j)\Big) \\ &\quad\times\prod_{1\leqslant i,j\leqslant n}\exp\Big(-\mathrm{i}\arg(x_i - y_j)\Big)\end{aligned} \tag{4.19}$$

and recalling equations (4.4), (3.10) and (3.11), we obtain that

$$
\begin{aligned}
&\left\langle T(\Psi_{\mathrm{R}}^{*}(x_1)\cdots\Psi_{\mathrm{R}}(y_n))\right\rangle^{V} \\
&\quad = \Xi(V)^{-1}\int \mathcal{D}\Phi\ \exp\left(-\frac{1}{8\pi}(d\Phi, d\Phi)\right) \\
&\qquad \times \prod_{i=1}^{n}\left(\exp\left(\frac{\mathrm{i}}{2}\Phi(x_i)\right)\exp\left(\frac{\pi}{2}\Delta^{-1}(x_i,x_i)\right)\right) \\
&\qquad \times \prod_{j=1}^{n}\left(\exp\left(-\frac{\mathrm{i}}{2}\Phi(y_j)\right)\exp\left(\frac{\pi}{2}\Delta^{-1}(y_j,y_j)\right)\right) \\
&\qquad \times \frac{\prod_{1\leqslant i<i'\leqslant n}|x_i-x_{i'}|^{\frac{1}{2}}\prod_{1\leqslant j<j'\leqslant n}|y_j-y_{j'}|^{\frac{1}{2}}}{\prod_{1\leqslant i,j\leqslant n}|x_i-y_j|^{\frac{1}{2}}}\exp\left(\mathrm{i}\sum_{1\leqslant i<i'\leqslant n}\arg(x_i-x_{i'})\right) \\
&\qquad \times \exp\left(\mathrm{i}\sum_{1\leqslant j<j'\leqslant n}\arg(y_j-y_{j'})\right)\exp\left(-\mathrm{i}\sum_{1\leqslant i,j\leqslant n}\arg(x_i-y_j)\right) \\
&\qquad \times \exp\left(-\frac{1}{8\pi^2}\left({}^{*}(d\Phi+\varphi_{\underline{x},\underline{y}}), V^{*}(d\Phi+\varphi_{\underline{x},\underline{y}})\right)\right) \\
&\quad = \Xi(V)^{-1}(2\pi)^{\frac{-n}{2}}\int \mathcal{D}\Phi\ \left\{\exp\left(-\frac{1}{8\pi}(d\Phi+\varphi_{\underline{x},\underline{y}}, d\Phi+\varphi_{\underline{x},\underline{y}})\right)\right\}_{\mathrm{ren}} \\
&\qquad \times \exp\left(-\frac{1}{8\pi^2}\left({}^{*}(d\Phi+\varphi_{\underline{x},\underline{y}}), V^{*}(d\Phi+\varphi_{\underline{x},\underline{y}})\right)\right) \\
&\qquad \times \prod_{i=1}^{n} :\exp\left(\frac{\mathrm{i}}{2}\left(\Phi(x_i)+\alpha_{\underline{x},\underline{y}}(x_i)\right)\right): \\
&\qquad \times \prod_{j=1}^{n} :\exp\left(-\frac{\mathrm{i}}{2}\left(\Phi(y_j)+\alpha_{\underline{x},\underline{y}}(y_j)\right)\right): \\
&\quad = (2\pi)^{-\frac{n}{2}}\left\langle D(\underline{x},\underline{y})\prod_{i=1}^{n}:\exp\left(\frac{\mathrm{i}}{2}\Phi(x_i)\right):\prod_{j=1}^{n}:\exp\left(-\frac{\mathrm{i}}{2}\Phi(y_j)\right):\right\rangle^{V}
\end{aligned}
\tag{4.20}
$$

where we used the formal definition of normal ordering, equation (4.3).

This is exactly the result of the standard Abelian bosonization, equation (4.11). Hence we proved that one can identify $b^{(0)}$ with the bosonic field Φ of Abelian bosonization.

5. Condensed matter systems

In this section we discuss some applications of bosonization via gauge forms to fermionic systems of condensed matter physics. Let us start by defining the 'scaling limit' $S^{*}(A)$ of an effective action $S(A)$. We replace the gauge potential A by a rescaled potential

$$
A^{(\lambda)} = \lambda^{-1}A\left(\frac{x}{\lambda}\right). \tag{5.1}
$$

The 'scaling limit' $S^*(A)$ is defined as the coefficient of the leading term in an asymptotic expansion of $S(A^{(\lambda)})$ around $\lambda = \infty$ (see [1] for a more complete discussion).

Similarly, given a function $f(x)$, we define its scaling limit, $f^*(x)$, as the coefficient of the leading term in an asymptotic expansion of $f(\lambda x)$ around $\lambda = \infty$.

Remarkably, all many-body systems of non-relativistic fermions that can be treated analytically seem to have the property that $S^*(A)$ is quadratic in A. Here we describe some examples.

F. We start with the free non-relativistic fermion gas. The Euclidean action of the system is given by

$$I(\Psi, \Psi^*) = \int d^{d+1}x \left\{ \Psi^* \partial_0 \Psi + \frac{1}{2m} (\nabla \Psi)^2 + \mu \Psi^* \Psi \right\} (x) \tag{5.2}$$

where m is the mass of the fermions and μ the chemical potential.

In d=1, the Fermi surface consists of only two points, $\pm k_\mathrm{F}$. Since the scaling limit is dominated by excitations with momenta close to the Fermi surface, one expands the momentum-space fermionic two-point Green function around the points $\pm k_\mathrm{F}$ on the Fermi surface, in order to calculate its scaling limit.

The result of this analysis is that, in the scaling limit, one can introduce quasiparticle fields Ψ_L, Ψ_L^* and Ψ_R, Ψ_R^* corresponding to the points k_F and k_F, respectively. Then Ψ_L, Ψ_L^* are the fields describing left-moving excitations, while Ψ_R, Ψ_R^* describe right movers. These excitation approximately obey a relativistic dispersion relation, $\omega \approx |p|$, where $\omega = E - E_\mathrm{F}$ and $p = k - k_\mathrm{F}$. The original field variable $\Psi(x)$ is related to the relativistic field variables $\Psi_\mathrm{L}(x)$ and $\Psi_\mathrm{R}(x)$ by

$$\Psi(x) \simeq \exp\left(-\mathrm{i}k_\mathrm{F}x^1\right) \Psi_\mathrm{R}(x) + \exp\left(\mathrm{i}k_\mathrm{F}x^1\right) \Psi_\mathrm{L}(x) . \tag{5.3}$$

The Euclidean action of Ψ_L and Ψ_R is given by

$$I(\Psi_\mathrm{R}, \Psi_\mathrm{R}^*, \Psi_\mathrm{L}, \Psi_\mathrm{L}^*) = \int d^2x \, \{\Psi_\mathrm{R}^*(\partial_0 + \mathrm{i}v_\mathrm{F}\partial_1)\Psi_\mathrm{R} + \Psi_\mathrm{L}^*(\partial_0 - \mathrm{i}v_\mathrm{F}\partial_1)\Psi_\mathrm{L}\}(x) . \tag{5.4}$$

The action (5.4) is the Euclidean action of free massless Dirac fermions, with the Fermi velocity v_F playing the role of the velocity of light. As a consequence the effective action $S^*(A)$ determined by the action (5.4) is quadratic in A; see section 4. Using the bosonization method of section 4, one obtains a quadratic action for the potential $b^{(0)} \equiv \Phi$ of the conserved $U(1)$-current, see equation (4.16). These features of one-dimensional systems have been known for many years [25].

More recently, it has been realized (see [26] for the original observations) that similar ideas on the scaling limit can be used in any dimension d. The sum over the two points of the Fermi surface in (5.3) must be replaced by an integration over a higher $(d-1)$-dimensional Fermi surface; see [27].

Let Ω be the $(d-1)$-dimensional unit sphere in momentum space. Elements of Ω are denoted by ω. The extension of formula (5.3) to higher dimensions in given by

$$\Psi(x) \simeq \int_\Omega d\omega \, \exp\left(\mathrm{i}k_\mathrm{F}\boldsymbol{x} \cdot \boldsymbol{\omega}\right) \Psi_\omega(x) \tag{5.5}$$

and the Euclidean action for the fields Ψ_ω, describing the scaling limit of the free Fermi gas can be given formally as

$$I(\{\Psi_\omega, \Psi_\omega^*\}) \simeq \int_\Omega d\omega \int d^{d+1}x \, \{\Psi_\omega^*(\partial_0 + \mathrm{i}v_\mathrm{F}\boldsymbol{\omega} \cdot \nabla) \exp\left(-\alpha(\boldsymbol{\omega} \wedge \nabla)^2\right) \Psi_\omega\}(x) . \tag{5.6}$$

where α is a positive constant.

It is shown in [2] that $S^\star(A)$ is obtained as the integral over the set of directions $\pm\,\omega$ in momentum space of contributions coming from the degrees of freedom described by the fields Ψ_ω and $\Psi_{-\omega}$, corresponding to antipodal points, $\pm\omega$, on the Fermi surface. Every such contribution just corresponds to the contribution of a one-dimensional free fermion system and is quadratic in $A_\omega \equiv (A_0, \omega\cdot A)$. Since $S^\star(A)$ is the integral of the one-dimensional actions $S^\star(A_\omega)$ over all pairs of points $\pm\omega$ in Ω, it, too, is quadratic in A. Technical details of this calculation will appear in [2]. These considerations yield an explicit expression for $S^\star(A)$. Let $\Pi^{\mu\nu}(x, y)$ denote the vacuum polarization tensor of a system of non-interacting fermions at zero temperature, and let $\rho^{(1)} = \rho dx^0$, where ρ is the density of the system, then:

$$S^\star(A) = \tfrac{1}{2}(A, \Pi^\star A) + \mathrm{i}(\rho^{(1)}, A)\,. \tag{5.7}$$

By invariance under $U(1)$-gauge transformations, translations, rotations and parity, the expression for $\Pi^{\mu\nu}$ is given, in momentum space, by

$$\begin{aligned}
\Pi^{ij}(k) &= \Pi_\perp(k)\left(\delta_{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right) + \Pi_{\|}(k)\frac{k^i k^j}{\boldsymbol{k}^2} \\
\Pi^{i0}(k) &= -\Pi_{\|}(k)\frac{k^i}{k_0} \\
\Pi^{00}(k) &\equiv \Pi_0(k) = \Pi_{\|}(k)\frac{\boldsymbol{k}^2}{k_0^2}
\end{aligned} \tag{5.8}$$

for $i, j = 1, \ldots, d$. The explicit form of $\Pi_0^\star$ in d=1 is given by

$$\Pi_0^\star(k) = \chi_0 \frac{v_F^2 k_1^2}{k_0^2 + v_F^2 k_1^2} \tag{5.9}$$

and, in higher dimensions, to leading order in $|v_F\boldsymbol{k}/k_0|$ and $|k_0/v_F\boldsymbol{k}|$, we have that

$$\begin{aligned}
\Pi_0^\star(k) &= \left(\chi_0 + \lambda_0\left|\frac{k_0}{v_F\boldsymbol{k}}\right|\right)\Theta\left(1 - \left|\frac{k_0}{v_F\boldsymbol{k}}\right|\right) + \tilde\chi_0\left|\frac{v_F\boldsymbol{k}}{k_0}\right|^2\Theta\left(\left|\frac{k_0}{v_F\boldsymbol{k}}\right| - 1\right) \\
\Pi_\perp^\star(k) &= \left(\chi_\perp \boldsymbol{k}^2 + \lambda_\perp\left|\frac{k_0}{v_F\boldsymbol{k}}\right|\right)\Theta\left(1 - \left|\frac{k_0}{v_F\boldsymbol{k}}\right|\right) + \tilde\chi_0\Theta\left(\left|\frac{k_0}{v_F\boldsymbol{k}}\right| - 1\right)
\end{aligned} \tag{5.10}$$

where $\chi_0, \lambda_0, \tilde\chi_0, \chi_\perp, \lambda_\perp$ are constants depending on d, m and v_F [29].

According to the argument given before, the expression for $\Pi^\star$ in $d > 1$ can be derived from the integration over ω of one-dimensional vacuum polarizations, Π_ω, relative to the 'quasiparticle fields' $\Psi_\omega, \Psi_{-\omega}$

$$(A, \Pi^\star A) = \tfrac{1}{2}\int_\Omega d\omega\; (A_\omega, \Pi_\omega A_\omega)\,.$$

Let us stress again that the quadratic nature of $S^\star(A)$ does not just follow from a 'small-A' approximation and dimensional analysis, but it is the result of explicit cancellations arising from the structure of the fermion two-point function in the scaling limit.

Remark 5.1. One can directly bosonize the scaling-limit action of the free Fermi gas expressed in terms of the 'quasi- particle fields' $\{\Psi_\omega, \Psi_\omega^*\}, \omega \in \Omega$, by introducing real scalar fields $\{\Phi_\omega\}, \omega \in \Omega$, identifying Φ_ω with $\Phi_{-\omega}$. The result is the Luther–Haldane bosonization [26, 28], in the Euclidean path-integral formalism. From section 4 one can derive an explicit expression for the fermionic fields Ψ_ω, Ψ_ω^* in the bosonized theory. With obvious notation, we have

$$\Psi_{\pm\omega}(x) \to (2\pi)^{-\frac{1}{4}} D_\omega(x, 1) : \exp\left(\pm\frac{\mathrm{i}}{2}\Phi_\omega(x)\right) :$$

$$\Psi_{\pm\omega}^*(x) \to (2\pi)^{-\frac{1}{4}} D_\omega(x, -1) : \exp\left(\pm\frac{\mathrm{i}}{2}\Phi_\omega(x)\right) :$$

From now on we omit the trivial term $\mathrm{i}(\rho^{(1)}, A)$ in the effective actions. This corresponds to redefining the density $\mathcal{J}^0(\Psi, \Psi^*)$ by subtracting the background density ρ. Furthermore we set $v_{\mathrm{F}} = 1$.

I. Insulators and incompressible quantum fluids form another class of fermionic systems whose (bulk) effective action in the scaling limit, $S^*(A)$, is quadratic in A. Here incompressibility means that the connected correlation functions of the current $\mathcal{J}^\mu(\Psi, \Psi^*)$ have cluster properties better than those encountered in systems whose large-scale physics is dominated by Goldstone bosons. From incompressibility it follows [1] that $S^*(A)$ is local, and, for systems, with translation, rotation and parity invariance, it is given by

$$S^*(A) = \tfrac{1}{2}\int d^{d+1}x \ \left\{g_{\mathrm{E}}(dA)^{0i}(dA)_{0i}(x) + g_{\mathrm{B}}(dA)^{ij}(dA)_{ij}(x)\right\} \tag{5.11}$$

where $g_{\mathrm{E}}, g_{\mathrm{B}}$ are constants.

H. The quantum Hall fluids are parity-breaking, two-dimensional incompressible systems. For Laughlin fluids with translation and rotation invariance, it has been shown in [1, 3] that S^* is the Chern–Simons action

$$S^*(A) = \frac{\mathrm{i}\sigma_{\mathrm{H}}}{4\pi}\int dA \wedge A \tag{5.12}$$

where σ_{H} is the Hall conductivity, $\sigma_{\mathrm{H}} = \frac{1}{2\ell+1}$, $\ell = 0, 1, 2, \ldots$.

S. London theory and computations based on perturbation theory suggest that $S^*(A)$ for BCS superconductors is also quadratic and is given by

$$S^*(A) = \tfrac{1}{2}\int d_x^{d+1} \left\{\frac{1}{\lambda_{\mathrm{L}}^2}(A^{\mathrm{T}})^2(x) + \int d^{d+1}y \ (A_0 - \partial_0\Delta_d^{-1}\partial_i A^i)(x)\Pi_s(x-y)(A_0 - \partial_0\Delta_d^{-1}\partial_j A^j)(y)\right\} \tag{5.13}$$

where λ_{L} is a constant (the London penetration depth)

$$A_i^{\mathrm{T}} = A_i - \partial_i\Delta_d^{-1}\partial^j A_j \tag{5.14}$$

with $i, j = 1, \ldots, d$, Δ_d denotes the d-dimension Laplacian and Π_s is the scaling limit of the scalar component of the vacuum polarization in the superconductor.

To leading order in $|k/k_0|$ and $|k_0/k|$, Π_s in $d > 1$ is given by

$$\Pi_s(k) = \frac{1}{\lambda_L^2}\left|\frac{k}{k_0}\right|^2 \Theta\left(\left|\frac{k_0}{k}\right| - 1\right) + \chi_s \Theta\left(1 - \left|\frac{k_0}{k}\right|\right) \tag{5.15}$$

where χ_s is a constant depending on d, m, v_F [30]. The scaling limit effective actions of the systems F, I, H, S given by equations (5.7) and (5.11)–(5.13) yield a bosonized action of the form

$$\tilde{S}^*(db) = \frac{1}{8\pi^2}\left({}^*db, (\Pi^*)^{-1}\, {}^*db\right) \tag{5.16}$$

where, in the notation of (5.8), Π^* is given by

$$\begin{aligned}
&\text{F:} && \text{equations (5.9), (5.10)} \\
&\text{I:} && \Pi_0^*(k) = g_E k^2 \qquad \Pi_\perp^*(k) = g_B k^2 + g_E k_0^2 \\
&\text{H:} && (\Pi^*)^{\mu\nu}(k) = \frac{i}{2\pi\sigma_H}\epsilon^{\mu\nu\rho}k_\rho \\
&\text{S:} && \Pi_0^*(k) = \Pi_s(k) \qquad \Pi_\perp^*(k) = \frac{1}{\lambda_L^2}.
\end{aligned} \tag{5.17}$$

If Π has the form (5.8) then one finds

$$\begin{aligned}
\tilde{S}^*(db) = \frac{1}{8\pi^2}\int d^{d+1}x\, d^{d+1}y \Big\{({}^*db)^i(x)\left[(\Pi_\perp^*)^{-\frac{1}{2}}(\delta_{ij} - \partial_i \Delta_d^{-1}\partial_j)(\Pi_\perp^*)^{-\frac{1}{2}}\right](x, y) \\
\times ({}^*db)^j(y)\Big\}.
\end{aligned} \tag{5.18}$$

In particular, in $d = 2$ b is a 1-form and (5.18) simplifies to

$$\tilde{S}^*(db) = \frac{1}{8\pi^2}\int d^{2+1}k\, \left\{(\Pi_\perp^*)^{-1}(k)k^2\left(b_0 - k_0\frac{k\cdot b}{k^2}\right)^2 + (\Pi_0^*)^{-1}(k)k^2(b^T)^2\right\}. \tag{5.19}$$

For Laughlin fluids, the dual action is given by

$$\tilde{S}^*(db) = -\frac{i}{4\pi\sigma_H}\int b \wedge db. \tag{5.20}$$

5.1. Duality in two-dimensional systems

Note that, in two space dimensions, A and b are one 1-forms, and from equations (5.7), (5.10), (5.11), (5.13), (5.16), (5.17), (5.19) and (5.20) it follows that the actions $S^*(A)$ and $\tilde{S}^*(db)$ are related by a remarkable 'duality':

$$\begin{aligned}
S^*(b)|_I &\propto \tilde{S}^*(db)|_S \\
S^*(b)|_S &\propto \tilde{S}^*(db)|_I \\
S^*(b)|_H &\propto \tilde{S}^*(db)|_H
\end{aligned} \tag{5.21}$$

with (g_E, g_B) corresponding to $(\lambda_L^2, \chi_S^{-1})$, and σ_H going to σ_H^{-1}.

In particular, since $S^*(A)|_I$ is the Maxwell action, it follows that $\tilde{S}^*(db)|_S$ describes a massless mode, the Goldstone boson of the superconducting state with broken gauge invariance, known as the Anderson–Bogoliubov mode. By the Kramers–Wannier duality, this mode can also be described by an angular variable which is a free field.

5.2. Disorder fields for Laughlin fluids

Some care is needed in defining the disorder fields whose expectation values are proportional to fermion Green functions of Hall fluids in the scaling limit. In fact, the naive definition guessed from (3.9), (3.10)

$$\left\langle D(\underline{x}, \underline{q})\right\rangle^* = \Xi^{-1} \int \mathcal{D}[b] \exp\left(\mathrm{i}\,\frac{\sigma_{\mathrm{H}}}{4\pi} \int (b + \tilde{\alpha}_{\underline{x},\underline{q}}) \wedge (db + \varphi_{\underline{x};\underline{q}})\right)$$
$$= \Xi^{-1} \int_{\mathcal{A}^1_{\underline{x};\underline{q}}} \mathcal{D}[\tilde{b}] \exp\left(\mathrm{i}\frac{\sigma_{\mathrm{H}}}{4\pi} \int \tilde{b} \wedge f(\tilde{b})\right),$$

does not make sense, since $\tilde{\alpha}_{\underline{x};\underline{q}}$ and $\tilde{b}$ are not well defined 1-forms on $M_{\underline{x}} \equiv \mathbb{R}^{2+1}\backslash\{\underline{x}\}$. In more mathematical terms, one observes that the definition of the Chern–Simons action for connections $\tilde{b}$ on a non-trivial bundle $\mathcal{A}^1$ (see the appendix) requires the specification of a reference connection $\tilde{b}_0 \in \mathcal{A}^1$. With $b = (\tilde{b} - \tilde{b}_0) \in \Lambda^1(M_{\underline{x}})$, the Chern–Simons action is given by [9]

$$S_{\mathrm{CS}}(b, \tilde{b}_0) = \int b \wedge db + 2b \wedge f(\tilde{b}_0). \tag{5.22}$$

(In what follows we adapt the procedure of [10], where a more detailed discussion can be found. With obvious modifications, our construction can be extended to more general Chern–Simons gauge theories.)

In order to define the disorder field proportional to a 2-point fermion function, we choose $\tilde{b}_0$ as follows: let $E^{\pm}$ denote a 2-form in $\mathbb{R}^2$ with support in a cone C , with apex at 0 and contained in the positive-(negative) time half-space, satisfying

$$dE^{\pm} = {}^*\delta_0 \tag{5.23}$$

and denote by E_x its translation by x.

Then, for $x_1, \ldots, x_2$ in the positive-time half-space and $x_{r+1}, \ldots, x_{2n}$ in the negative-time half-space, with $x_i^0 < x_{i+1}^0$, and for a set of charges, $q_1, \ldots, q_{2n}$ with $\sum_{i=1} q_i = 0$, $|q_i| = 1$, we define $\tilde{b}_0 \equiv \tilde{\alpha}_{\underline{x};\underline{q};\underline{E}}$ by

$$f(\tilde{\alpha}_{\underline{x};\underline{q};\underline{E}}) \equiv \varphi_{\underline{x};\underline{q};\underline{E}} = \sum_{i=1}^{r} q_i E_x + \sum_{j=r+1}^{2n} q_j E \tag{5.24}$$

and assume that the cones $\{C_{\underline{x}}\}$ 'join at infinity', (see [10]). Since $M_{\underline{x}}$ has a boundary, $\{\underline{x}\}$, $S_{\mathrm{CS}}(b, \tilde{b}_0)$ is not gauge-invariant and we need a compensating term in the action to restore gauge invariance; this term is obtained by adding to $\varphi_{\underline{x};\underline{q};\underline{E}}$ a two-form $j_{\underline{x};\underline{q}}$ which is a sum of dual currents with support in a set of straight lines joining each point x_i to its projection on to the time-zero plane, $(0, \boldsymbol{x}_i)$ and carrying charge q_i. The expectation value of the disorder field, $D(\underline{x}, \underline{q}, \underline{E})$, is defined by

$$\left\langle D(\underline{x}, \underline{q}, \underline{E})\right\rangle^*$$
$$= \begin{cases} \Xi^{-1} \displaystyle\int \mathcal{D}[b]\ \exp\left\{\mathrm{i}\frac{\sigma_{\mathrm{H}}}{4\pi}\left[S_{\mathrm{CS}}(b, \tilde{\alpha}_{\underline{x};\underline{q};\underline{E}}) + \int 2b \wedge j_{\underline{x};\underline{q}}\right]\right\} & \text{if } d(j_{\underline{x};\underline{q}} + \varphi_{\underline{x};\underline{q};\underline{E}}) = 0 \\ 0 & \text{otherwise}. \end{cases} \tag{5.25}$$

One can think the support of $j_{\underline{x};\underline{q}}$ as representing the Euclidean worldlines of static fermions created (destroyed) at the boundary points x_i where $q_i = -1$ $(+1)$; since the electric flux is conserved by gauge invariance, the electric flux lines spread at the end points $\underline{x}$ in shapes described by the distributions $E_{\underline{x}}$.

The result of integration in (5.25) is the following: let $\alpha_{\underline{x};\underline{q};\underline{E}}$ be a 1-form satisfing

$$d\alpha_{\underline{x};\underline{q};\underline{E}} = j_{\underline{x};\underline{q}} + \varphi_{\underline{x};\underline{q};\underline{E}} \tag{5.26}$$

then

$$\left\langle D(\underline{x}, \underline{q}, \underline{E})\right\rangle^{\star} = \exp\left(\mathrm{i}\frac{\pi}{\sigma_{\mathrm{H}}} S_{\mathrm{CS}}(\alpha_{\underline{x};\underline{q};\underline{E}})\right) . \tag{5.27}$$

If the cones $\{C_{\underline{x}}\}$ are shrunk to non-intersecting paths $\{\gamma_{\underline{x}}\}$ and we denote the corresponding 1-form by $\alpha_{\underline{x};\underline{q};\underline{\gamma}}$ instead of $\alpha_{\underline{x},\underline{q},\underline{E}}$, then $d\alpha_{\underline{x};\underline{q};\underline{\gamma}}$ is dual to an oriented link and $S_{\mathrm{CS}}(\alpha_{\underline{x};\underline{q};\underline{\gamma}})$ gives the Gauss linking number of that link.

From this observation one deduces that, for a Laughlin fluid with $\sigma_{\mathrm{H}} = 1/(2\ell + 1)$, $\ell = 0, 1, 2, \ldots$, the particles described by the Green functions $\left\langle D(\underline{x},\underline{q}, \underline{E})\right\rangle^{\star}$ are fermions, because if two arguments, x_i and x_j, with $q_i = q_j$ are interchanged by a smooth deformation of the paths γ_{x_i} and γ_{x_j} then $\left\langle D(\underline{x}, \underline{q}, \underline{E})\right\rangle^{\star}$ changes sign.

6. Adding perturbations: some applications

Let us perturb the 'reference' systems F–S by a density–density and/or current–current interaction described by a perturbation term in the action, $I_{\mathrm{pert}}(\Psi, \Psi^{*})$ given by equation (2.22) with $V = (V_{\mu\nu})$ positive-definite. Furthermore let us make the following perturbative assumption.

Assumption P. The scaling limit of the perturbed theory coincides with the perturbation of the scaling limit of the original theory.

Adopting definition (2.19), the bosonized action of the perturbed theory in the scaling limit is given by

$$\tilde{S}^{\star}_{\mathrm{tot}}(db) = \frac{1}{8\pi^2}({}^{*}db, (\Pi^{\star})^{-1} + V^{\star}){}^{*}db) \tag{6.1}$$

provided asssumption P holds. From (5.18) one derives, for example, that the two-point function of charge density and current in the scaling limit is given by

$$\langle \mathcal{J}_{\mu}(\Psi, \Psi^{*}; x)\mathcal{J}_{\nu}(\Psi, \Psi^{*}; y)\rangle^{\star} = 4\pi^2 \left((\Pi^{\star})^{-1} + V^{\star}\right)^{-1}_{\mu\nu}(x, y) . \tag{6.2}$$

Perturbing the free theory, F, equation (5.19) reproduces the result of the Euclidean RPA approximation in the scaling limit. This proves that, under asssumption P, the RPA approximation gives the (leading term in the) scaling limit of density and current correlation functions. Using the form (5.6) of the action for a system of free fermions, one can argue that asssumption P holds for two-dimensional systems, provided there are no interaction between electrons at antipodal points of the Fermi surface, i.e. provided that the 'Cooper channel' is turned off.

6.1. Plasmon gap and Anderson–Higgs mechanism

Equation (6.2) is useful for understanding the phenomenon of 'mass generation' via Coulomb repulsion: if one perturbs a free Fermi gas or a BCS superconductor in two or more dimensions by (d-dimensional) repulsive Coulomb two-body interactions, with V given by $\hat{V}_{00}(\boldsymbol{k}) = e^2|\boldsymbol{k}|^{-2}$, and $V_{\mu\nu} = 0$ otherwise, then the Fourier transform of the density two-point function has quasiparticle poles at $k_0 = \pm iM$, $M > 0$, as $|\boldsymbol{k}| \searrow 0$. (For perturbations described by a two-body potential $V_{00}(\boldsymbol{x})$ decaying faster than the Coulomb potential, the denominator of the two point function in Fourier transform vanishes, as $\boldsymbol{k}, k_0 \searrow 0$).

For a perturbation of the free theory, F, M is the *plasmon gap*. The fact that M is strictly positive, has been interpreted in [31] as the result of a 'generalized Goldstone theorem' applicable in the presence of Coulomb forces.

For superconductor,a similar phenomenon has first been analysed by Anderson [32]. From (6.2) and (5.10), (5.15), and for $V_{00} = e^2/|\boldsymbol{k}|^2$ one obtains

$$\left(\langle \mathcal{J}^0(\Psi, \Psi^*; k)\mathcal{J}^0(\Psi, \Psi^*; -k)\rangle^V\right)^* \underset{|\boldsymbol{k}\backslash k_0|\to 0}{\sim} \frac{4\pi^2}{c}\left|\frac{k_0}{\boldsymbol{k}}\right|^2 + \frac{e^2}{|\boldsymbol{k}|^2} = \frac{4\pi^2|\boldsymbol{k}|^2}{c(k_0^2 + M^2)} \tag{6.3}$$

where $M = e^2/c$, $c = \tilde{\chi}_0^{-1}$ for F, $c = \lambda_L^{-2}$ for S.

As remarked above for the perturbations of a system of free electrons satisfying asssumption P, the RPA approximation is exact in the scaling limit. This explains why the RPA value of the plasmon gap coincides with the exact value obtained by a non-perturbative analysis in [31].

For the superconducting theory one can go one step further then if one uses a form for $\Pi_0^* \equiv \Pi_s$ that correctly interpolates between the behaviour at small $|\boldsymbol{k}/k_0|$ and at large $|\boldsymbol{k}/k_0|$ described in (5.15), namely

$$\Pi_s = \frac{\boldsymbol{k}^2}{\lambda_L^2 k_0^2 + \chi_s^{-1}\boldsymbol{k}^2}. \tag{6.4}$$

The vacuum polarization tensor $\Pi^{\mu\nu}$ defined in (5.8), with $\Pi_\perp = \Pi_\perp^*$ and $\Pi_0 = \Pi_0^* = \Pi_s$ as in (5.17), (5.15) describes a superconductor in the scaling limit as system of non-interacting, massless $U(1)$ Goldstone bosons. Taking asssumption P for granted, we now wish to study the effect of two-body Coulomb repulsion on the quasiparticle spectrum of a superconductor.

Let us first do the calculation for a two-dimensional system. By equation (5.19) the action $\tilde{S}_{\rm tot}^*(db)$ is then given by

$$\tilde{S}_{\rm tot}^*(db) = \frac{1}{8\pi^2}(b, \tilde{\Pi}b) \tag{6.5}$$

where $\tilde{\Pi}^{\mu\nu}$ is given by (5.8), with

$$\tilde{\Pi}_\perp = \lambda_L^2|\boldsymbol{k}|^2\chi_s^{-1}k_0^2 + e^2 \qquad \tilde{\Pi}^{00} = \lambda_L^2\boldsymbol{k}^2. \tag{6.6}$$

From these formulae we learn that $b^{\rm T}$ describes a *massive* quasiparticle, with a dispersion relation given by

$$\omega(\boldsymbol{k}) = \sqrt{\chi_s(\lambda_L^2|\boldsymbol{k}|^2 + e^2)}. \tag{6.7}$$

These formulae can easily be generalized to systems in d dimensions, with $d > 2$. The result is the same: the theory describes one massive quasiparticle with a dispersion relation given by (6.7). This can be seen by recalling (6.2) with $V^{\mu\nu}(k) = \delta^{\mu 0}\delta^{\nu 0}e^2/\boldsymbol{k}^2$.

The phenomenon described above is the Anderson–Higgs mechanism.

6.2. The existence of the charge operator

Another issue that can be analysed, using equation (6.2), is the existence of the charge operator $\hat{Q}$ (see section 3). Consider a Wilson loop of rank $d-1$, $W_\alpha(\mathcal{L}^R_{(d-1)})$, with $\mathcal{L}^R_{(d-1)}=\partial\Sigma^R_{(d)}$. Then $W_\alpha(\mathcal{L}^R_{(d-1)})$ determines an operator proportional to $\exp(\mathrm{i}\alpha\hat{Q}(\Sigma^R_{(d)}))$, where $\hat{Q}(\Sigma^R_{(d)})$ measures the charge contained in the region $\Sigma^R_{(d)}$; see equation (3.37).

Using equation (6.1) one obtains

$$\begin{aligned}\langle W_\alpha(\mathcal{L}^R_{(d-1)})\rangle^\star &= \exp\left(-\frac{\alpha^2}{2}\left\langle\int_{\Sigma^R_{(d)}} db(x)\int_{\Sigma^R_{(d)}} db(y)\right\rangle^\star\right)\\ &= \exp\left(-\frac{\alpha^2}{2}(2\pi)^2\int_{\Sigma^R_{(d)}} db^\mu(x)\int_{\Sigma^R_{(d)}} db^\nu(y)\left((\Pi^\star)^{-1}+V^\star\right)^{-1}_{\mu\nu}(x,y)\right)\\ &= \exp\left(-2\pi^2\alpha^2\int_{\Sigma^R_{(d)}} db^0(x)\int_{\Sigma^R_{(d)}} db^0(y)\left((\Pi^\star)^{-1}+V^\star\right)^{-1}_{00}(x,y)\right). \end{aligned} \tag{6.8}$$

For $V=0$, straightforward computation gives

$$\langle W_\alpha(\mathcal{L}^R_{(d-1)})\rangle^\star \underset{R\to\infty}{\sim} \begin{cases}\exp\left(-c|\mathcal{L}^R_{(d-1)}|\ln R\right) & \text{for F}\\ \exp\left(-c|\mathcal{L}^R_{(d-1)}|\right) & \text{for I}\\ 1 & \text{for H}\\ \exp\left(-c|\mathcal{L}^R_{(d-1)}|\ln R\right) & \text{for S}\end{cases} \tag{6.9}$$

for some positive constants (denoted by c).

Hence, according to the criterion of section 3, the charge operator $\hat{Q}$ does not exist for free systems of fermions and for superconductors. Charge density fluctuations are so strong that the limit (3.29) does not exist. For insulators or quantum Hall fluids, equation (6.9) implies instead the existence of the charge operator which, at zero temperature, defines a superselection rule.

According to the 't Hooft duality one expects that the two-point disorder correlation functions of the bosonized theory, for systems F and S, have at most power law decay in spatial directions, while, for systems I and H, they have at least exponential decay in spatial directions. Since the fermion fields Ψ, Ψ^* are proportional to disorder fields, one infers that, for systems I and H, the two-point fermion Green functions exhibit at least exponential decay in spatial directions.

If we add a short-range density–density perturbation, $V_{00}(\boldsymbol{k}) \simeq$ constant > 0, for $|\boldsymbol{k}| \approx 0$, then under asssumption P the results do not change for the perturbations of systems F, I, S; for perturbed quantum Hall fluids, we obtain a perimeter decay for $\langle W_\alpha(\mathcal{L}^R_{(1)})\rangle^\star$.

If we add a long-range density–density perturbation, with $V_{00}(\boldsymbol{k}) = g|\boldsymbol{k}|^{-\alpha}, g > 0$, $0 < \alpha \leqslant 2$, we obtain perimeter decay also for the perturbed systems of F and S.

6.3. The orthogonality catastrophe

In this last subsection we discuss in some detail an application of our formalism to the problem of the 'orthogonality catastrophe' for static sources in a Landau–Fermi liquid perturbed by a repulsive density–density interactions. 'Orthogonality catastrophe' just means that the ground state of the system is orthogonal to the ground state in the presence of a static source, i.e. their overlap vanishes [33].

For $d > 1$, we have the following heuristic picture: the injection of a static source triggers the production of a number, divergent in the thermodynamic limit, of particle–hole pairs of arbitrarily low energy near the Fermi surface, and this leads to the orthogonality catastrophe. An infinite number of particle–hole pairs is produced because there are infinitely many degrees of freedom in the vicinity of the Fermi surface. Hence in one-dimensional systems, a different mechanism must be responsible for the orthogonality catastrophe. In fact, for d=1, the density–density interaction drives the system away from Landau liquid behaviour, and the orthogonality catastrophe can be related to the vanishing of the wave-function renormalization characteristic of Luttinger liquids [34].

We now show that under asssumption P our formalism leads to a clean proof of the 'orthogonality catastrophe' in all dimensions. Our proof shows that the term responsible for a vanishing overlap is largely insensitive to the structure of the density–density interaction in $d > 1$, but it strongly depends on the behaviour of the interaction, as the momentum $|\boldsymbol{k}| \searrow 0$, in d=1.

We start by analysing how one can express the overlap between the ground states in the path-integral formalism.

Let $H(J)$ be the Hamiltonian of the fermionic system in the presence of a static source J. Assume that the bottom of the spectrum of $H(J)$ is given by an eigenvalue, $E(J)$, corresponding to the energy of the ground state, $|0\rangle_J$ in the presence of the static source. Denote by $|0\rangle$ the ground state of the Hamiltonian H of the fermion system. Then we have that

$$\lim_{t\to\infty} \langle 0| \exp\left(-t\,(H(J) - E(J))\right) |0\rangle = |\langle 0|0\rangle_J|^2 . \tag{6.10}$$

This suggest that

$$\lim_{t\to\infty} \frac{\langle 0|\mathrm{e}^{-tH(J)}|0\rangle}{\langle 0|\mathrm{e}^{-2tH(J)}|0\rangle^{\frac{1}{2}}} = |\langle 0|0\rangle_J| \tag{6.11}$$

and, indeed, equation (6.11) can be proved, provided the limit on the l.h.s. exists.

A standard application of the Feynman–Kac formula proves the equality

$$\langle 0|\mathrm{e}^{-tH(J)}|0\rangle = \Xi(J_t) \tag{6.12}$$

where $\Xi(J_t)$ is the (grand-canonical) partition function of the system in the presence of a current described by a 1-form J_t given by

$$J_t(x) = \begin{cases} J\delta(\boldsymbol{x})dx^0 & 0 \leqslant x^0 \leqslant t \\ 0 & \text{otherwise}. \end{cases} \tag{6.13}$$

Equation (6.11) then gives

$$|\langle 0|0\rangle_J| = \lim_{t\to\infty} \frac{\Xi(J_t)}{\Xi(J_{2t})^{\frac{1}{2}}} . \tag{6.14}$$

Using an interaction term of the form (2.22) we conclude from equations (2.23) (with the notation of section 3) and assumption P that

$$\Xi(J_t) = \int \mathcal{D}\Psi\,\mathcal{D}\Psi^*\ \exp\left\{-\left[S(\Psi,\Psi^*) + \left((\mathcal{J}(\Psi,\Psi^*) - J_t), V(\mathcal{J}(\Psi,\Psi^*) - J_t)\right)\right]\right\}$$

and the scaling limit of $\Xi(J_t)$ is given by

$$\Xi(J_t)^* = \int \mathcal{D}[b]\ \exp\left(-\tfrac{1}{2}\left(\frac{{}^*db}{2\pi}, (\Pi^*)^{-1}\frac{{}^*db}{2\pi}\right)\right)$$
$$\times \exp\left(-\tfrac{1}{2}\left(\left(\frac{{}^*db}{2\pi}+J_t\right), V^*\left(\frac{{}^*db}{2\pi}+J_t\right)\right)\right)$$
$$= \exp\left(-\tfrac{1}{2}\left(J_t, V^*((\Pi^*)^{-1}+V^*)^{-1}(\Pi^*)^{-1}J_t\right)\right). \tag{6.15}$$

We now specialize (6.15) to a potential V describing a time-independent, rotation-invariant density–density interaction. The Fourier transform of V is then given by

$$V^*_{\mu\nu}(k) = \delta_{\mu\nu}\delta_{\alpha 0}V_0(|\boldsymbol{k}|). \tag{6.16}$$

We assume that

$$V_0(|\boldsymbol{k}|) \sim |\boldsymbol{k}|^{-\alpha} \tag{6.17}$$

with $0 \leqslant \alpha < 2$. Inserting equations (6.16) and (6.13) into (6.15), we obtain that

$$\Xi(J_t)^* = \exp\left\{-\frac{J^2}{2}\int_0^t dx^0 \int_0^t dy^0\ (\Pi_0^* + V_0^{-1}\ )^{-1}(x^0-y^0, 0)\right\}. \tag{6.18}$$

Using the explicit form of Π_0^*, equation (5.10), in $d > 1$, we conclude that

$$\frac{-\ln \Xi(J_t)^*}{J^2} = \int_0^t d\tau \int_0^\tau ds \int_{|\boldsymbol{k}|<\Lambda} \frac{d^d k}{(2\pi)^d}\int_0^{|\boldsymbol{k}|}\frac{dk_0}{2\pi}\frac{\cos k_0 s}{\chi_0 + V_0^{-1}(|\boldsymbol{k}|)+\lambda_0|k_0/\boldsymbol{k}|}$$
$$+\int_0^t d\tau \int_0^\tau ds \int_0^\Lambda \frac{dk_0}{2\pi}\int_{|\boldsymbol{k}|<k_0}\frac{d^d k}{(2\pi)^d}\frac{\cos k_0 s}{\tilde{\chi}_0(k^2/k_0^2)+V_0^{-1}(|\boldsymbol{k}|)} \tag{6.19}$$

where Λ is some ultraviolet cutoff.

The second term in (6.19) is easily bounded uniformly in t. In the first term we define

$$\gamma \equiv \frac{k_0}{|\boldsymbol{k}|} \qquad \chi_0(|\boldsymbol{k}|) \equiv \chi_0 + V_0^{-1}(|\boldsymbol{k}|) \tag{6.20}$$

and we rewrite it as

$$\frac{1}{2\pi}\int_0^t d\tau \int_0^1 \frac{d\gamma}{\gamma}\int_{|\boldsymbol{k}|<\Lambda}\frac{d^d k}{(2\pi)^d}\frac{\sin|\boldsymbol{k}|\gamma\tau}{\chi_0(|\boldsymbol{k}|)+\lambda_0\gamma}$$
$$= \frac{1}{2\pi}\int_0^t d\tau \int_{|\boldsymbol{k}|<\Lambda}\frac{d^d k}{(2\pi)^d}\frac{1}{\chi_0(|\boldsymbol{k}|)}\left\{\sin\left(\frac{\chi_0(|\boldsymbol{k}|)|\boldsymbol{k}|\tau}{\lambda_0}\right)\right.$$
$$\times\left[\mathrm{ci}\left(|\boldsymbol{k}|\tau\left(\frac{\chi_0(|\boldsymbol{k}|)+1}{\lambda_0}\right)\right) - \mathrm{ci}\left(\frac{|\boldsymbol{k}|\tau\chi_0(|\boldsymbol{k}|)}{\lambda_0}\right)\right] - \cos\left(\frac{\chi_0(|\boldsymbol{k}|)|\boldsymbol{k}|\tau}{\chi_0}\right)$$
$$\left.\times\left[\mathrm{si}\left(|\boldsymbol{k}|\tau\left(\frac{\chi_0(|\boldsymbol{k}|)}{\lambda_0}+1\right)\right) - \mathrm{si}\left(\frac{|\boldsymbol{k}|\tau\chi_0(|\boldsymbol{k}|)}{\lambda_0}\right)\right] + \mathrm{si}(|\boldsymbol{k}|\tau) + \frac{\pi}{2}\right\}$$
$$\underset{t\to\infty}{\sim} \frac{1}{2\pi}\int_{|\boldsymbol{k}|<\Lambda}\frac{d^d k}{(2\pi)^d}\frac{1}{\chi_0(|\boldsymbol{k}|)}\left\{\frac{\lambda_0}{\chi_0(|\boldsymbol{k}|)|\boldsymbol{k}|}\left[\ln\left(\frac{\chi_0(|\boldsymbol{k}|)+\lambda_0}{\chi_0(|\boldsymbol{k}|)|\boldsymbol{k}|t} - C\right)\right]\right.$$
$$\left. + t\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{t|\boldsymbol{k}|}\right)\right\} \tag{6.21}$$

where ci and si are the cosine integral and the sine integral functions, respectively. Hence, as $t \nearrow \infty$, we have a contribution linear in t, a contribution logarithmic in t and a finite correction. According to equations (6.14), (6.19) and (6.21), the overlap in $d > 1$ is given by

$$|\langle 0|0\rangle_J| \sim \lim_{t\to+\infty} \exp\left[-\frac{|\lambda_0|J^2}{4\pi}\int_{|\boldsymbol{k}|<\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{|\boldsymbol{k}|}\frac{1}{(\chi_0+V_0^{-1}(|\boldsymbol{k}|))^2}\ln t\right] = 0\,. \tag{6.22}$$

Equation (6.22) proves the orthogonality catastrophe in $d > 1$. If one analyses where the vanishing of the overlap (6.22) comes from, one realizes that it is due to the term

$$\lambda_0\left|\frac{k_0}{\boldsymbol{k}}\right|\Theta\left(1-\left|\frac{k_0}{\boldsymbol{k}}\right|\right) \tag{6.23}$$

in $\Pi_0^\star$, i.e. it originates from the region in the spectrum (low k_0, low $|\boldsymbol{k}|$, $k_0 < |\boldsymbol{k}|$), where particle–hole excitations near the Fermi surface dominate the large-scale physics. If one omits the term (6.23), one can check that the overlap $\langle 0|0\rangle_J$ becomes finite. The term in $\Xi(J_t)^\star$ responsible for the vanishing overlap in $d > 1$ is essentially independent of the specific structure of the interaction $V_0(|\boldsymbol{k}|)$ at low $|\boldsymbol{k}|$, since, to leading order in $|\boldsymbol{k}|$, $V_0^{-1}(|\boldsymbol{k}|)$ gives a contribution negligible with respect to χ_0, for an interaction with $\alpha > 0$

Next, we discuss the overlap in one dimension. Using the expression for Π_0 in d=1, (equation (5.9)), we obtain

$$\begin{aligned}
\frac{-\ln\Xi(J_t)^\star}{J^2} &= \int_0^t d\tau\int_0^\tau ds\int_{-\Lambda}^{\Lambda}\frac{dk_1}{2\pi}\int_{-\Lambda'}^{\Lambda'}\frac{dk_0}{2\pi}\frac{(k_0^2+k_1^2)\cos k_0 s}{(k_0^2+k_1^2)V_0^{-1}(k_1)+\chi_0 k_1^2}\\
&= \int_0^t d\tau\int_{-\Lambda}^{\Lambda}\frac{dk_1}{2\pi}\int_{-\Lambda'}^{\Lambda'}\frac{dk_0}{2\pi}\frac{\sin k_0\tau}{k_0}\left[V_0(k_1)-\frac{\chi_0 V_0^2(k_1)k_1^2}{k_0^2+k_1^2(1+\chi_0 V_0(k_1))}\right]\\
&\sim_{\Lambda'\nearrow\infty}\int_{-\Lambda}^{\Lambda}\frac{dk_1}{2\pi}\int_0^t d\tau\bigg[\tfrac{1}{2}V_0(k_1)\\
&\qquad -\tfrac{1}{2}\frac{\chi_0 V_0^2(k_1)}{1+\chi_0 V_0(k_1)}\left(1-\exp\left(-|k_1|(1+\chi_0 V_0(k_1))^{\frac{1}{2}}t\right)\right)\bigg]\\
&= \int_{-\Lambda}^{\Lambda}\frac{dk_1}{2\pi}\tfrac{1}{2}\frac{V_0(k_1)}{1+\chi_0 V_0(k_1)}t\\
&\qquad +\tfrac{1}{2}\frac{\chi_0 V_0^2(k_1)}{(1+\chi_0 V_0(k_1))^{\frac{3}{2}}}\frac{1}{|k_1|}\left(1-\exp\left(-|k_1|(1+\chi_0 V_0(k_1))^{\frac{1}{2}}t\right)\right)\,.
\end{aligned} \tag{6.24}$$

From equations (6.24) and (6.14) it follows that:

$$\begin{aligned}
|\langle 0|0\rangle_J| \sim \lim_{t\to\infty}\exp\bigg[-J^2\int_{-\Lambda}^{\Lambda}\frac{dk_1}{2\pi}\frac{\chi_0 V_0^2(k_1)}{(1+\chi_0 V_0(k_1))^{\frac{3}{2}}}\frac{1}{|k_1|}\\
\times\bigg\{\tfrac{1}{2}\left(1-\exp\left(-|k_1|(1+\chi_0 V_0(k_1))^{\frac{1}{2}}t\right)\right)\\
-\tfrac{1}{4}\left(1-\exp\left(-|k_1|(1+\chi_0 V_0(k_1))^{\frac{1}{2}}2t\right)\right)\bigg\}\bigg]\,.
\end{aligned} \tag{6.25}$$

In contrast to the result in $d > 1$, the way the limit (6.25) approaches 0 as $t \nearrow \infty$ depends on the specific structure of $V_0(k_1)$. For example, if

$$V_0(k_1) \underset{k_1 \to 0}{\sim} g|k_1|^{-\alpha} \qquad 1 > \alpha > 0 \quad g > 0 \tag{6.26}$$

then

$$|\langle 0|0\rangle_J| \sim \lim_{t\to\infty} \exp\left(-J^2 c(\alpha) g^{\frac{1}{2-\alpha}} \chi_0^{\frac{\alpha-1}{2-\alpha}} t^{\frac{\alpha}{2-\alpha}}\right) = 0 \tag{6.27}$$

where $c(\alpha)$ is a positive constant. If

$$V_0(k_1) \underset{k_1 \to 0}{\sim} g > 0 \tag{6.28}$$

then

$$|\langle 0|0\rangle_J| \sim \lim_{t\to\infty} \exp\left(-\frac{J^2}{4\pi}\frac{\chi_0 g^2}{(1+\chi_0 g)^{\frac{3}{2}}} \ln t\right) = 0\,. \tag{6.29}$$

The behaviour (6.29) is characteristic of the Luttinger liquid [34], and, in general, (6.25) suggests a non-Fermi liquid character of the system.

Remark 6.1. If, instead of using the true Π_0 given in equation (5.9), we use a $\tilde{\Pi}_0$ obtained by extending the form of Π_0 in d=1 to higher dimensions, i.e.

$$\tilde{\Pi}_0(k) = \chi_0 \frac{|\boldsymbol{k}|^2}{k_0^2 + |\boldsymbol{k}|^2} \tag{6.30}$$

then one recovers the results of [35].

Remark 6.2. If, in $d > 1$, we compute the partition function, $\Xi(J_t^v)$, of the system in the presence of a current, J_t^v, describing the motion of a source with constant (Euclidean) velocity $\boldsymbol{v}$ in a fermion system with transverse current–current interactions, given by:

$$V^\star_{\mu\nu}(k) = \begin{cases} \left(\delta_{ij} - \dfrac{k_i k_j}{\boldsymbol{k}^2}\right) V_\perp(k_0, |\boldsymbol{k}|) & i, j = 1, \ldots, d \\ 0 & \text{otherwise} \end{cases} \tag{6.31}$$

with $V_\perp(0) \neq 0$, then, using (6.15), we obtain that

$$\Xi(J_t^v)^\star = \exp\left\{-\frac{J^2}{2}\int_0^t dx^0 \int_0^t dy^0 \int \frac{d^{d+1}k}{(2\pi)^{d+1}} \left(\boldsymbol{v}^2 - \frac{(\boldsymbol{v}\cdot\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\right)\left(\Pi_\perp^\star(k) + V_\perp^{-1}(k)\right)^{-1} \right.$$
$$\left. \times \exp\left(\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v} + k_0)(x^0 - y^0)\right)\right\}. \tag{6.32}$$

To leading order in $|k_0/\boldsymbol{k}|$, for $|k_0/\boldsymbol{k}| < 1$, we derive from (5.10) that

$$\Pi_\perp(k) = \chi_\perp |\boldsymbol{k}|^2 + \lambda_\perp \left|\frac{k_0}{\boldsymbol{k}}\right| \qquad \Pi_0(k) = \chi_0 + \lambda_0 \left|\frac{k_0}{\boldsymbol{k}}\right|.$$

Hence, for $k_0 = 0$,, $\boldsymbol{k} \searrow 0$, $\Pi_\perp$ vanishes as $|\boldsymbol{k}|^2$,whereas Π_0 remains finite. As a consequence, in contrast to density–density interactions current–current interactions in $d > 1$ may significantly change the behaviour of $\Xi(J_t^v)^\star$, suggesting that Fermi systems in $d > 1$ with long range current–current interactions could show non-Fermi liquid behaviour.

In a two-dimensional system with Coulomb–Ampère current–current interactions, a non-Fermi liquid ('Luttinger') behaviour has been exhibited within the eikonal approximation [36] (see also [37]).

Appendix. Gauge forms

In this appendix, we introduce the concept of gauge forms of rank k, or generalized $U(1)$-connections of rank k [7, 8]. For k=1, it coincides with the standard concept of a $U(1)$-gauge field or, in mathematical language, of a $U(1)$-connection.

Definition. Let M be an open subset of $\mathbb{R}^{d+1}$ and let $\mathcal{U} = \{U_i\}_{i\in I}$ denote a covering of M by open subsets. Then a gauge form of rank $k, \tilde{a}^{(k)}$, is a collection of k-forms $\{a_i^{(k)}\}_{i\in I}$ on $\mathcal{U}$ such that
(1) for $x \in U_i \cap U_j$

$$a_i^{(k)}(x) - a_j^{(k)}(x) = z_{ij}^{(k)}(x)2\pi \tag{A.1}$$

with $z_{ij}^{(k)} \in \Lambda^k(U_i \cap U_j)$ a closed form with integral periods (i.e. its integral over an arbitrary closed $k-$surface contained in $U_i \cap U_j$ is an integer). From (A.1) it follows that

$$da_i^{(k)}(x) = da_j^{(k)}(x) \tag{A.2}$$

for x in $U_i \cap U_j$ and hence $\{da_i^{(k)}\}_{i\in I}$ defines a closed $(k+1)$-form, $f^{(k+1)}(\tilde{a})$, on M.
(2) For every closed $(k+1)$-dimensional surface, $S_{(k+1)}$, in M, $f^{(k+1)}(\tilde{a})$ has the property

$$\frac{1}{2\pi}\int_{S_{(k+1)}} f^{(k+1)}(\tilde{a}) \in \mathbb{Z}\,. \tag{A.3}$$

The form $f^{(k+1)}(\tilde{a})$ is called the curvature (or field strength) of $\tilde{a}^{(k)}$. The space of gauge forms of rank k is denoted by $\mathcal{A}^k$.

Remark A.1. From the definition it follows that a gauge form of rank $k > 1$ can be seen as a generalization of a $U(1)$-connection (called a generalized $U(1)$-connection of rank k) on a principal bundle, $\mathcal{P}^k$, whose transition functions are given by the $(k-1)$-forms $\{\lambda_{ij}^{(k-1)}\}$. This bundle is called a $U(1)$-fibre bundle of rank k, and $\mathcal{A}^k \equiv \mathcal{A}^k(\mathcal{P}^k)$ is the space of $U(1)$-connections of rank k on $\mathcal{P}^k$. By equation (A.11), the difference of two connections, $\tilde{a}^{(k)}, \tilde{a}'^{(k)} \in \mathcal{A}^k(\mathcal{P}^k)$, is a globally defined k-form, i.e. $\tilde{a}^{(k)} - \tilde{a}'^{(k)} \in \Lambda^k(M)$. Hence, $\mathcal{A}^k$ is an affine space modeled on $\Lambda^k(M)$ i.e.

$$\mathcal{A}^k = \tilde{a}^{(k)} + \Lambda^k(M) \qquad \tilde{a}^{(k)} \in \mathcal{A}^k. \tag{A.4}$$

Two $U(1)$-fibre bundles of rank $k, \mathcal{P}^k, \mathcal{P}'^k$ characterized by the transition functions $\{\lambda_{ij}\}, \{\lambda'_{ij}\}$, are said to be isomorphic if

$$\lambda_{ij}'^{(k-1)} = \lambda_{ij}^{(k-1)} + \lambda_i^{(k-1)} - \lambda_j^{(k-1)} + \zeta_{ij}^{(k-1)}2\pi \tag{A.5}$$

for $\lambda_i^{(k-1)} \in \Lambda^{k-1}(U_i), \lambda_j^{(k-1)} \in \Lambda^{k-1}(U_j)$, and $\zeta_{ij}^{(k-1)} \in \Lambda^{k-1}(U_i \cap U_j)$ is a closed form with integral periods. An isomorphism class of $U(1)$-fibre bundles of rank k is called a $U(1)$-bundle of rank k. If M is k-connected, the $U(1)$-bundles of rank k are classified by $H^{k+1}(M, \mathbb{Z})$, which is isomorphic to the subgroup of $H^{k+1}_{\mathrm{deR}}(M)$ given by the cohomology classes of $(k+1)$-forms of integral periods. The classification map associates to a bundle the cohomology class of $\frac{1}{2\pi} f^{(k+1)}(\tilde{a})$, where $\tilde{a}^{(k)}$ is a connection on the bundle [8].

The space of gauge forms, $\mathcal{A}^k$, carries an action of the gauge group $\mathcal{G}^{k-1}$, whose elements, $\tilde{\lambda}^{(k-1)}$, are collections $\{\lambda_i^{(k-1)}\}_{i\in I}$ of (k-1)-forms, with the following patching property:

$$\lambda_i^{(k-1)}(x) - \lambda_j^{(k-1)}(x) = \zeta_{ij}^{(k-1)}(x)2\pi \tag{A.6}$$

for $x \in U_i \cap U_j$, with $\zeta_{ij}^{(k-1)}$ closed of integral periods.

The action of $\mathcal{G}^{k-1}$ on $\mathcal{A}^k$ is given by

$$a_i^{(k)} \longrightarrow a_i^{(k)} + d\lambda_i^{(k-1)} . \tag{A.7}$$

Note that if $H^k_{\text{deR}}(M) = 0$, then

$$\mathcal{G}^{k-1} \simeq \Lambda^{k-1}(M) \tag{A.8}$$

i.e. $\tilde{\lambda}^{(k-1)}$ is determined by a globally defined (k-1)-form, $\lambda^{(k-1)}$. Furthermore if $H^{k+1}_{\text{deR}}(M) = 0$, or, more generally, if the cohomology class of $f^{k+1}(\tilde{a})$ is zero, then

$$\mathcal{A}^k \simeq \Lambda^k(M) \tag{A.9}$$

i.e. one can view $\tilde{a}^{(k)}$ as a globally defined k-form $a^{(k)}$.

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